THE LAGRANGIAN:

$$\rho \frac{D}{Dt} \cdot \vec{u} = \rho \left(\frac{\partial}{\partial t} \vec{u} + (\vec{u} \cdot \vec{\nabla}) \vec{u} \right) = \vec{f}$$

force density field

Total time derivative / Convective time derivative

Now let's consider a volume V enclosing some fluid with spatially varying mass density ρ and velocity $\vec{u}$.

So total change in momentum over the volume V can be written as a sum of forces acting on the volume and the momentum flux the surface of V.

$$\frac{d}{dt} \int_V \rho \vec{u} \, d\vec{r} = - \oint_S \rho \vec{u} \, \underbrace{\vec{u} \cdot \hat{n}} \, ds + \int_V \vec{f} \, d\vec{r}$$

surface normal

force density

Velocity component normal to the integration surface.

$\rho \vec{u} \rightarrow$ momentum density.

$\vec{\nabla} f$ = surface normal of scalar f (A vector)

$\vec{\nabla} \vec{f}$ = surface normal of vector $\vec{f}$ (A tensor)

DETAILED

$\vec{\nabla} \vec{u} \longrightarrow$ Surface normal to $\vec{u}$

$\hookrightarrow \vec{u} \cdot \hat{n}$

REMEMBER!

It's operating on a vector not a scalar. But we can simplify the situation comparing with a scalar operation.

$$\rho \left(\frac{\partial}{\partial t} \vec{u} + \vec{u} \cdot \nabla \vec{u} \right) = \vec{f}$$

$$\Rightarrow \rho \left(\frac{\partial}{\partial t} \vec{u} \right) = -\rho \vec{u} \cdot \vec{\nabla} \vec{u} + \vec{f}$$

$$\Rightarrow \underbrace{\frac{d}{dt} (\rho \vec{u})}_{\text{Volume element}} = - \underbrace{\rho \vec{u} \vec{u} \cdot \hat{n}}_{\text{Surface element}} + \underbrace{\vec{f}}_{\text{Volume element}}$$

$$\Rightarrow \frac{d}{dt} \int_V \rho \vec{u} \, d\vec{r} = - \oint_S \rho \vec{u} \vec{u} \cdot \hat{n} \, ds + \int_V \vec{f} \, d\vec{r}$$