Dispersion of Compressional Alfvén Waves

$$\frac{\partial}{\partial t}\vec{B}_1 = \vec{\nabla}\times(\vec{u}_1\times\vec{B}_0) \quad --- \; 1$$

$$\frac{\partial}{\partial t}\rho_1 + \rho_0(\vec{\nabla}\cdot\vec{u}_1) = 0 \quad --- \; 2$$

$$\rho_0\frac{\partial}{\partial t}\vec{u}_1 = -\vec{\nabla}p_1 + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}_1)\times\vec{B}_0 \quad -- \; 3$$

$$p_1 = \frac{KT}{M}\rho_1 \quad - \; 4$$

Rewriting the above expressions in Fourier space,

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$$1) \rightarrow -i\omega\vec{B}_1 = i\vec{K}\times(\vec{u}_1\times\vec{B}_0) = i(\vec{K}\cdot\vec{B}_0)\vec{u}_1 - i(\vec{K}\cdot\vec{u}_1)\vec{B}_0 \quad -- \; 1.1$$

$$2) \rightarrow -i\omega\rho_1 + i\rho_0\vec{K}\cdot\vec{u}_1 = 0 \quad -- \; 2.1$$

$$3) \rightarrow -i\omega\rho_0\vec{u}_1 = -i\vec{K}p_1 + \frac{i}{\mu_0}(\vec{K}\times\vec{B}_1)\times\vec{B}_0$$

$$\Rightarrow -i\omega\rho_0\vec{u}_1 = -i\vec{K}p_1 - \frac{i}{\mu_0}\left[(\vec{B}_0\cdot\vec{B}_1)\vec{K} - (\vec{K}\cdot\vec{B}_0)\vec{B}_1\right]$$

$$\Rightarrow \rho_0\omega\vec{u}_1 = \vec{K}p_1 + \frac{1}{\mu_0}\left[(\vec{B}_0\cdot\vec{B}_1)\vec{K} - (\vec{K}\cdot\vec{B}_0)\vec{B}_1\right] \quad -- \; 3.1$$

$$4) \rightarrow p_1 = \frac{KT}{M}\rho_1 = c_s^2\rho_1 \quad --- \; 4.1$$

Simplifying the above expressions,

$$1.1 \Rightarrow B_1 = \frac{(\vec{K}\cdot\vec{u}_1)\vec{B}_0 - (\vec{K}\cdot\vec{B}_0)\vec{u}_1}{\omega} \quad -- \; 1.2$$

$$2.1 \Rightarrow \rho_1 = \rho_0\frac{\vec{K}\cdot\vec{u}_1}{\omega} \quad ---- \; 2.2$$

$$4.1 \Rightarrow p_1 = c_s^2\rho_0\frac{\vec{K}\cdot\vec{u}_1}{\omega} \quad ---- \; 4.2$$

Now using (1.2), (2.2), and (4.1) in (3.1)

$$\rho_0 \omega \vec{u}_1 = \vec{k} c_s^2 \rho_0 \frac{\vec{k}\cdot\vec{u}_1}{\omega} + \frac{1}{\mu_0}\left[\vec{B}_0 \cdot \left\{\frac{(\vec{k}\cdot\vec{u}_1)\vec{B}_0 - (\vec{k}\cdot\vec{B}_0)\vec{u}_1}{\omega}\right\}\vec{k} - (\vec{k}\cdot\vec{B}_0)\left\{\frac{(\vec{k}\cdot\vec{u}_1)\vec{B}_0 - (\vec{k}\cdot\vec{B}_0)\vec{u}_1}{\omega}\right\}\right]$$

$$\Rightarrow \rho_0 \omega^2 \vec{u}_1 = \vec{k} c_s^2 \rho_0 (\vec{k}\cdot\vec{u}_1) + \frac{1}{\mu_0}\Big[(k\cdot u_1) B_0^2 \vec{k} - (\vec{k}\cdot\vec{B}_0)(\vec{u}_1\cdot\vec{B}_0)\vec{k} - (\vec{k}\cdot\vec{B}_0)(\vec{k}\cdot\vec{u}_1)\vec{B}_0 + (\vec{k}\cdot\vec{B}_0)^2 \vec{u}_1\Big]$$

$$\Rightarrow \omega^2 \vec{u}_1 = \vec{k} c_s^2 (\vec{k}\cdot\vec{u}_1) + \frac{1}{\mu_0 \rho_0}\Big[(k\cdot u_1) B_0^2 \vec{k} - (\vec{k}\cdot\vec{B}_0)(\vec{u}_1\cdot\vec{B}_0)\vec{k} - (\vec{k}\cdot\vec{B}_0)(\vec{k}\cdot\vec{u}_1)\vec{B}_0 + (\vec{k}\cdot\vec{B}_0)^2 \vec{u}_1\Big]$$

$$\Rightarrow \left[\omega^2 - \frac{(\vec{k}\cdot\vec{B}_0)^2}{\mu_0 \rho_0}\right]\vec{u}_1 = \left\{\left(c_s^2 + \frac{B_0^2}{\mu_0 \rho_0}\right)\vec{k} - \frac{\vec{k}\cdot\vec{B}_0}{\mu_0 \rho_0}\vec{B}_0\right\}(\vec{k}\cdot\vec{u}_1) - \frac{(\vec{k}\cdot\vec{B}_0)(\vec{u}_1\cdot\vec{B}_0)}{\mu_0 \rho_0}\vec{k} \quad — (5)$$

We also have $V_A = \sqrt{\dfrac{B_0^2}{\mu_0 \rho_0}}$

Assuming the equilibrium magnetic field B_0 is along z direction, and wave vector $\vec{K}$ lies in x-z plane. Let θ be the angle between $\vec{B}_0$ and $\vec{K}$, then equ. (5) can be reduced to eigen value equ.

$$\begin{pmatrix} \omega^2 - K^2 V_A^2 - K^2 C_s^2 \sin^2\theta & 0 & -K^2 C_s^2 \sin\theta\cos\theta \\ 0 & \omega^2 - K^2 V_A^2 \cos^2\theta & 0 \\ -K^2 C_s^2 \sin\theta\cos\theta & 0 & \omega^2 - K^2 C_s^2 \cos^2\theta \end{pmatrix} \begin{pmatrix} u_{1x} \\ u_{1y} \\ u_{1z} \end{pmatrix} = 0$$

The solubility condition for the above equation demands the determinant of the square matrix is zero.

This condition yields,

$$(\omega^2 - K^2 V_A^2 \cos^2\theta)\left[\omega^4 - \omega^2 K^2 (V_A^2 + C_s^2) + K^4 V_A^2 C_s^2 \cos^2\theta\right] = 0$$

Rearranging,

$$\left(\frac{\omega^4}{K^4} - \frac{\omega^2}{K^2}(C_s^2 + V_A^2) + C_s^2 V_A^2 \cos^2\theta\right)\left(\frac{\omega^2}{K^2} - V_A^2 \cos^2\theta\right) = 0$$

For the limiting case $T = 0 \rightarrow C_s = 0$

The the equ. simplifies to

$$\left(\frac{\omega^4}{K^4} - \frac{\omega^2}{K^2} V_A^2\right)\left(\frac{\omega^2}{K^2} - V_A^2 \cos^2\theta\right) = 0$$