Plasma as a mixture of charged gases

The MHD description of plasma considers the motion of charged particles (i.e. electrons and ions) as a single fluid. In this particular lecture, we are going to break that assumption and take separate equations for individual species for describing the dynamics.

Multi-component plasmas

Assumptions:

1) Fully ionized plasmas
2) High conductivity (electron moves freely)
3) Two component plasma (i.e. electron and ion)

NOTE: In multicomponent plasmas, apart from electron there can be multiple ion species and even ions with different electric charges.

Now, we can write the continuity and momentum eqns. for different species:

CONTINUITY EQN:

$$\frac{\partial}{\partial t} n_e + \vec{\nabla}\cdot(n_e \vec{u}_e) = 0 \rightarrow \text{(1A)}$$

$$\frac{\partial}{\partial t} n_i + \vec{\nabla}\cdot(n_i \vec{u}_i) = 0 \rightarrow \text{(1B)}$$

NOTE: Notice the difference. Here we are using number density instead of mass density.

MOMENTUM EQN: $(e>0)$

$$m_e n_e \left(\frac{\partial}{\partial t} \vec{u}_e + \vec{u}_e \cdot \vec{\nabla} \vec{u}_e \right) = -\vec{\nabla} p_e - e n_e \left[\vec{E} + \vec{u}_e \times \vec{B} \right] + \vec{P}_{e \to i} \rightarrow 2A$$

$$m_i n_i \left(\frac{\partial}{\partial t} \vec{u}_i + \vec{u}_i \cdot \vec{\nabla} \vec{u}_i \right) = -\vec{\nabla} p_i + e n_i \left[\vec{E} + \vec{u}_i \times \vec{B} \right] - \vec{P}_{i \to e} \rightarrow 2B$$

$\vec{P}$ = momentum transfer from electrons to ions through collisions.

We assume that the momentum exchange can be written as,

$$\vec{P} = \vartheta \, n_e n_i \left(\vec{u}_i - \vec{u}_e \right) \rightarrow 3$$

$\vartheta > 0$ (proportionality const.)

ARGUMENT:

$\vec{P}$ should be a vector in the direction of the relative velocity $(\vec{u}_i - \vec{u}_e)$ and it should vanish if the velocities are equal.

$\vec{P}$ should also vanish if one of the densities become zero.

Summary: if $u_i = u_e$ $\rightarrow$ no collision

if $n_{ie} \to 0$ $\rightarrow$ no collision

From our early lectures, we have the expressions for collision frequency as,

$$\nu_{e,i} = n\sigma v$$

Now, momentum transfer per collision $m|\vec{u}_i - \vec{u}_e|$

and number of collision per second per volume $\nu_{e,i} \cdot n_e$

So, momentum transfer per second per volume (momentum transfer rate)

$$P \sim \nu_{e,i}\, n_e\, m|\vec{u}_i - \vec{u}_e| \rightarrow (4)$$

Using (3) and (4) we can estimate the ϑ as,

$$\vartheta \sim \frac{\nu_{e,i}\, m_e}{n_i}$$

We also have the expression for plasma conductivity as:

$$\underline{\sigma} \approx \frac{n e^2}{m\, \nu_{e,i}}$$

which implies,

$$\underline{\sigma} \sim \frac{e^2}{\vartheta}$$

EQN. OF STATE:

$$p_e = f_e(n_e) \rightarrow \text{(5A)}$$

$$p_i = f_i(n_i) \rightarrow \text{(5B)}$$

Now, we will make an attempt to combine this multicomponent description of plasma to get back our original MHD description using few assumptions.

LET'S INTRODUCE FEW KNOWN QUANTITIES

MASS DENSITY (BULK MASS)

$$\rho = m_e n_e + m_i n_i \approx m_i n_i \quad \because m_i \gg m_e \rightarrow \text{(6)}$$

AVERAGE VELOCITY (OF BULK MASS)

$$\vec{u}\left(m_e n_e + m_i n_i\right) = n_e m_e \vec{u_e} + n_i m_i \vec{u_i}$$

$$\Rightarrow \vec{u} = \frac{n_e m_e \vec{u_e} + n_i m_i \vec{u_i}}{m_e n_e + m_i n_i} \rightarrow \text{(7)}$$

CHARGE DENSITY

$$\xi = e(n_i - n_e) \rightarrow \text{(8)}$$

CURRENT DENSITY

$$\vec{J} = e\left(n_i \vec{u_i} - n_e \vec{u_e}\right) \rightarrow \text{(9)}$$

Now, we are going to use the quantities to transform our two-component description.

1) ADD 1A and 1B [multiply 1A by m_e and 1B by m_i]

OPERATION: $(1A \times m_e + 1B \times m_i)$

$$\frac{\partial}{\partial t}\big(\underbrace{m_i n_i + m_e n_e}_{\rho}\big) + \vec{\nabla}\cdot\big(\underbrace{m_e n_e \vec{u}_e + m_i n_i \vec{u}_i}_{\vec{u}\rho}\big) = 0$$

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

CONTINUITY EQN FOR MHD

2) SUBSTRACT 1A and 1B [multiply them by e]

OPERATION: $(1A \times e - 1B \times e)$

$$\frac{\partial}{\partial t}\xi + \vec{\nabla}\cdot\vec{J} = 0$$

CHARGE CONTINUITY EQN

Now, combining momentum eqns. are a bit difficult because of the non-linear term (convective term). The simplest approach is to drop the non-linear terms assuming small perturbation to the static equilibrium (similar to dynamic solns.)

3) ADD 2A and 2B

$$\frac{\partial}{\partial t}\big[\underbrace{m_e n_e \vec{u}_e + m_i n_i \vec{u}_i}_{\rho\vec{u}}\big] = -\vec{\nabla}(p_e+p_i) + \overbrace{e(n_i-n_e)}^{\xi}\vec{E}$$

$$+ \underbrace{e(n_i\vec{u}_i - n_e\vec{u}_e)}_{\vec{J}}\times\vec{B} + \cancelto{0}{(P_{e\to i}-P_{i\to e})}$$

$$\Rightarrow \rho\frac{\partial\vec{u}}{\partial t} = -\vec{\nabla}(p_e+p_i) + \xi\vec{E} + \vec{J}\times\vec{B}$$

If we look carefully the previous equ. the pressure is not exactly different from our MHD description. In equ. of state, pressure is expressed in terms of temperature and the temperature is in fact a sum of T_i and T_e.

$$p_e + p_i = KT_e n_e + KT_i n_i \approx K(T_e + T_i)n$$

4) SUBSTRACT 2A and 2B [divide by m_e and m_i and multiply both by e]

OPERATION: $\left[(2B/m_i) \times e - (2A/m_e) \times e\right]$

$$\frac{\partial}{\partial t} \underbrace{e\left(n_i \vec{u_i} - n_e \vec{u_e}\right)}_{\vec{J}} = \vec{\nabla}\left[\frac{e}{m_e} p_e - \frac{e}{m_i} p_i\right] + \left[\frac{n_e}{m_e} e^2 + \frac{n_i e^2}{m_i}\right]\vec{E} + \left[\frac{n_e e^2}{m_e} \vec{u_e} + \frac{n_i e^2}{m_i} \vec{u_i}\right] \times \vec{B} - \left(\frac{e}{m_e} + \frac{e}{m_i}\right)\vec{P} \rightarrow (10)$$

FEW ASSUMPTIONS

1) pressure due to individual species is approximately similar.

$p_e \sim p_i$

2) charge to mass ratio for ions is small.

$\frac{e}{m_i} \rightarrow$ small ; $n_i \approx \frac{\rho}{m_i}$

$\rho = m_e n_e + m_i n_i$

$\rho \approx m_i n_i \because m_i \gg m$

We can also express, $n_e \approx \frac{\rho}{m_i} - \frac{\xi}{e}$ where, $\xi = e(n_i - n_e)$

3) m_e is small but flux $m_e\vec{u}$ can be large.

$$\frac{\partial}{\partial t}\vec{J} = \vec{\nabla}\left[\frac{e}{m_e}p_e - \cancel{\frac{e}{m_i}p_i}\right] + \left[\frac{n_e}{m_e}e^2 + \cancel{\frac{n_i e^2}{m_i}}\right]\vec{E}$$
$$+ \left[\frac{n_e e^2}{m_e}\vec{u}_e + \cancel{\frac{n_i e^2}{m_i}}\vec{u}_i\right]\times\vec{B} - \left(\frac{e}{m_e} + \cancel{\frac{e}{m_i}}\right)\vec{P}$$

using $\frac{e}{m_i} \to$ small approximation.

$$\frac{\partial}{\partial t}\vec{J} = \frac{e}{m_e}\vec{\nabla}p_e + \frac{n_e}{m_e}e^2\vec{E} + \frac{e^2}{m_e}n_e\vec{u}_e\times\vec{B} - \frac{e}{m_e}\vec{P} \rightarrow (11)$$

we have, $\vec{u}(m_e n_e + m_i n_i) = n_e m_e\vec{u}_e + n_i m_i\vec{u}_i$

$$\Rightarrow \vec{u}(m_i n_i) = n_e m_e\vec{u}_e + n_i m_i\vec{u}_i$$

$$\Rightarrow \vec{u} = \vec{u}_e n_e m_e\cdot\frac{1}{m_i n_i} + \vec{u}_i \rightarrow (12)$$

We also have, $\vec{J} = e(n_i\vec{u}_i - \vec{n}_e\vec{u}_e)$

$$\Rightarrow \vec{u}_i = \frac{\vec{J}}{e n_i} + \frac{n_e}{n_i}\vec{u}_e \rightarrow (13\ A)$$

$$\Rightarrow \vec{u}_e = -\frac{\vec{J}}{e n_e} + \frac{n_i\vec{u}_i}{n_e} \rightarrow (13\ B)$$

Using (13 A) in (12)

$$\Rightarrow \quad \vec{u} = \vec{u}_e n_e m_e \cdot \frac{1}{m_i n_i} + \frac{\vec{J}}{e n_i} + \frac{n_e}{n_i} \vec{u}_e$$

$$\boxed{\Rightarrow \quad \vec{u}_e = \frac{n_i}{n_e} \vec{u} - \frac{\vec{J}}{n_e e}}$$

ignoring $O(m_e/m_i)$

$\rightarrow$ (14 A)

Similarly, using (13B) in (12)

$$\Rightarrow \quad \vec{u} = \frac{n_e m_e}{m_i n_i} \left(\frac{n_i}{n_e} \vec{u}_i - \frac{\vec{J}}{e n_e} \right) + \vec{u}_i$$

$$= \frac{m_e}{m_i} \vec{u}_i - \frac{\vec{J} m_e}{m_i n_i e} + \vec{u}_i$$

$$= \frac{m_e}{m_i} \vec{u}_i - \frac{\vec{J} m_e}{\rho e} + \vec{u}_i$$

$$= \vec{u}_i - \frac{\vec{J} m_e}{\rho e}$$

ignoring $O(m_e/m_i)$

$$\boxed{\Rightarrow \quad \vec{u}_i = \vec{u} + \frac{\vec{J} m_e}{\rho e}}$$

$\rightarrow$ (14B)

Now, we can express, $\frac{n_e e^2}{m_e} \vec{u}_e$ as,

$$= \frac{n_e}{m_e} e^2 \left[\frac{n_i}{n_e} \vec{u} - \frac{\vec{J}}{n_e e} \right]$$

$$= \frac{e^2 n_i}{m_e} \vec{u} - \frac{e \vec{J}}{m_e}$$

$$\Rightarrow \frac{e^2\rho}{m_e m_i}\vec{u} - \frac{e}{m_e}\vec{J} \rightarrow (15)$$

Using (15) in (11)

$$\frac{\partial}{\partial t}\vec{J} = \frac{e}{m_e}\vec{\nabla}p_e + \frac{n_e e^2}{m_e}\vec{E} + \left(\frac{\rho e^2}{m_e m_i}\vec{u} - \frac{e}{m_e}\vec{J}\right)\times\vec{B} - \frac{e}{m_e}\vec{P} \rightarrow (16)$$

Now, using Quasi-neutrality we can write,

$$\xi\vec{E} \rightarrow 0 \qquad \xi \rightarrow \text{vanishes}$$

Then, $n_e \approx \frac{\rho}{m_i} = n_i$, $\xi \rightarrow$ small

So, $\vec{P}$ at scales $> \lambda_D$ (Quasi-neutrality)

$$-\frac{e}{m_e}\vec{P} = -\frac{e}{m_e}\vartheta n_e n_i[\vec{u_i} - \vec{u_e}]$$

$$= -\vartheta\frac{\rho}{m_i m_e} e n_e [u_i - u_e]$$

$$= -\vartheta\frac{\rho}{m_e m_i}\vec{J}$$

$$= -\frac{e^2\rho}{\sigma m_e m_i}\vec{J} \rightarrow (17)$$

Finally, we can write our MODIFIED OHM'S LAW

From equ. 16 using 17 and rearranging we get,

$$\underbrace{\frac{\partial}{\partial t}\vec{J}}_{\text{NON LOCAL IN TIME}} = \underbrace{\frac{e}{m_e}\vec{\nabla}p_e}_{\text{NON LOCAL IN SPACE}} + \frac{\rho e^2}{m_e m_i}\left(\vec{E} - \frac{1}{\underline{\sigma}}\vec{J} + \vec{u}\times\vec{B}\right) - \underbrace{\frac{e}{m_e}\vec{J}\times\vec{B}}_{\text{HALL CURRENT}} \quad \rightarrow 18$$

THIS MODIFIED OHM'S LAW CAN BE TURNED BACK INTO OUR OLD KNOWN FORM IF WE MAKE A TRANSITION TO SLOW AND LARGE SCALES

$$\frac{\partial}{\partial t}\vec{J} \rightarrow 0 \quad , \quad \vec{\nabla}p_e \rightarrow 0$$

So, FIRST and SECOND TERM becomes zero. But we still have the $\vec{J}\times\vec{B}$ term. In order to understand this term we need to revisit,

$$\frac{E}{B} \sim \omega\mathcal{L}$$

where, ω → characteristic frequency

and $\mathcal{L}$ → characteristic length scale in E-M fluctuation.

In MHD, the general consideration is the flow and electromagnetic field interacts strongly. In particular the flow can be represented as,

$$\boxed{U \sim \omega \mathcal{L}}$$

where, U is characteristic flow velocity

Now, the basic MHD assumption states that.

$$\frac{\epsilon_0 \mu_0 \left|\frac{\partial E}{\partial t}\right|}{|\vec{\nabla} \times \vec{B}|} \sim \frac{\omega}{c^2} \frac{E}{B} \mathcal{L} \sim \left(\frac{\omega \mathcal{L}}{c}\right)^2 \ll 1$$

which gives, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

Therefore, $$\vec{J} = \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B})$$

$$\sim \frac{B}{\mu_0 \mathcal{L}}$$

For large scale, $\mathcal{L} \to \infty$ then we can ignore the Hall term.

Finally, we can get back,

$$\boxed{\vec{J} = \sigma (\vec{E} + \vec{u} \times \vec{B})}$$

We can also find the different component of J w.r.t. the magnetic field.

$$m_e \frac{\partial}{\partial t} J_{\parallel} = e \frac{\partial}{\partial z} p_e + \frac{\rho e^2}{m_i}\left(E_{\parallel} - \frac{1}{\sigma} J_{\parallel}\right)$$

From the generalized OHM's law we can find the respective orders of magnitude by dividing each term with $\frac{E \rho e^2}{m_e m_i}$ and using $J \sim \frac{B}{\mu \mathcal{L}}$

$$\left(\frac{c}{\omega \mathcal{L}}\right)^2 \left(\frac{\omega}{\omega_{pe}}\right)^2 = \left(\frac{c_s}{\omega \mathcal{L}}\right)^2 \left(\frac{\omega}{\Omega_{ci}}\right) : 1 : \frac{\epsilon_0 \omega}{\sigma} \left(\frac{c}{\omega \mathcal{L}}\right)^2 : 1 : \left(\frac{c}{\omega \mathcal{L}}\right)^2 \frac{\Omega_{ci}\, \omega}{\omega_{pe}^2} \frac{m_i}{m}$$

Now, each term in equ. (18) can analyzed as follows,

- $\frac{\partial}{\partial t} \vec{J}$ → can be neglected $\left(\frac{c}{u}\right)^2 << \left(\frac{\omega_{pe}}{\omega}\right)^2$ i.e. low frequency
  where, $u = \omega \mathcal{L}$
- HALL TERM → can be neglected $\left(\frac{c}{u}\right)^2 << \left(\frac{m_e}{m_i}\right)\left(\frac{\omega_{pe}^2}{\Omega_{ci}\, \omega}\right)$ → Strict CONDITION
- pressure → can be neglected $\left(\frac{c_s}{u}\right)^2 << \frac{\Omega_{ci}}{\omega}$
- IDEAL MHD : $\left(\frac{c}{u}\right)^2 << \frac{\sigma}{\omega \epsilon_0}$ (high conductivity)

ALL INEQUALITIES REQUIRE: $\omega \to 0$ (low freq.)
$\alpha \to \infty$ (large scales)

So that, $\dfrac{c}{\omega \alpha} = \text{const.}$