MHD (MAGNATO-HYDRODYNAMICS)

Relatively simple but useful model for the dynamics of fully ionized plasmas considering the plasma as a medium which follows the basics laws of continuum (equ. of continuity, momentum equ.) in addition to strong electromagnetic force.

Example: Earth's upper ionosphere, magnetosphere, solar plasma.

Assumptions:

Single fluid description

- Conducting fluid/magnetized continuum ($N_p \gg 1$)
  implies → large conductivity ($\sigma \propto N_p \cdot \omega_p$)

- Valid when plasma is collision dominated
  implies → distribution functions are locally Maxwellian
  (collision dominated charged species)

  → time scale "τ" large in MHD to allow collision. much larger than the time light takes to travel through the medium.

  implies → Displacement current in Maxwell's equs can be neglected.

  → length scale "$\mathcal{L}$" is much larger than the mean free path between collisions.

  '$\mathcal{L}$' is also larger than "λ_D"
  → fluid is quasi-neutral (i.e. $n_i = n_e = n$)
  which gives single fluid description

→ length scale 'L' >> r_{Lo} (larmor radius)

neglect

electron diamagnetism and Hall effects

in the electron momentum equ.

and electron inertia can be incorporated in

the ion momentum equ.

which allows

single fluid description

- Charged species have the same temperature

→ Combining with quasi-neutrality

pressure will be same

→ energy equilibrium time is short compared

to characteristic time scale "τ".

IDEAL MHD

→ High Conductivity

(1) CONTINUITY EQN. (NO SOURCE OR SINK)

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\vec{u}\rho) = 0$$

(2) MOMENTUM EQN. (FORCE EQN.)

$$\underbrace{\rho\left(\frac{\partial}{\partial t}\vec{u} + \vec{u}\cdot\vec{\nabla}\vec{u}\right) = \underbrace{-\vec{\nabla}p}_{\text{Pressure}}}_{\text{NAVIER STOKES}} + \underbrace{\vec{J}\times\vec{B}}_{\text{magnetic force}} + \underbrace{\rho\vec{g}}_{\text{gravitational forces (solar plasma)}}$$

Space time varying currents are important for the dynamics of the plasma because of very high conductivity.

IMPORTANT THING TO NOTICE:

NO CONTRIBUTION from the electric field due to charge separation.

$$\vec{F} = q\vec{E} + \vec{V}\times\vec{B} \rightarrow \text{single particle}$$

For fluid,

$$(\text{Force density})\ \vec{f} = \cancelto{0}{\underbrace{\rho_q}_{\text{charge density}}\vec{E}} + \vec{J}\times\vec{B} \rightarrow \text{quite important}$$

This appears due to charge separation.

Due to high conductivity and large time scale it becomes negligible as it becomes quasi-neutral.

Now, we need to close these set of equations and find the expression for ρ, $\vec{J}$, $\vec{B}$ and their space-time variations.

WHY POISSON'S EQN DOESN'T HELP MUCH HERE?

Assuming quasi-neutrality, $n_i = n_e$

poisson's equ. $\nabla^2\phi = -\frac{e}{\epsilon_0}(n_i - n_e)$

$= 0 \rightarrow$ At large scales

(3) FARADAY'S LAW

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

Rotation of electric field equals to rate of change of magnetic field.

(4) AMPERE'S LAW:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \cancelto{0}{\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

$\rightarrow$ Displacement Current

we neglect it in MHD

BUT HOW?

Let's assume,

the characteristic length scale in MHD = $\mathcal{L}$ wavelength of plane wave

and " " time scale " " = τ period of harmonic oscillator

Similarly, characteristic amplitude fluctuations in electric field = $\tilde{E}_\perp$

" " " " magnetic field = $\tilde{B}$

Now using the characteristic quantities back in Ampere's Law we get,

Normalized form

$$\boxed{\vec{\nabla}' \times \vec{B}' = \mu_0 \vec{J} \frac{\mathcal{L}}{\tilde{B}} + \left(\frac{1}{c^2} \frac{\tilde{E}_\perp}{\tilde{B}} \frac{\mathcal{L}}{\tau}\right) \frac{\partial}{\partial t'} \vec{E}'}$$

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

LET'S FIND THE DETAILS:

$$\nabla \rightarrow \frac{\partial}{\partial x} \rightarrow \frac{1}{\Delta x} \rightarrow \frac{1}{\mathcal{L}}$$

$$\frac{\partial}{\partial t} \rightarrow \frac{1}{\Delta t} \rightarrow \frac{1}{\tau}$$

$$\vec{B}' = \frac{\vec{B}}{\tilde{B}}$$

$$\vec{E}' = \frac{\vec{E}}{\tilde{E}_\perp}$$

$\dfrac{\tilde{E}_\perp}{\tilde{B}} \rightarrow$ characteristic velocity $\left(\frac{\omega}{k}\right)$

$\dfrac{\mathcal{L}}{\tau} \rightarrow$ characteristic velocity

Assuming the planar wave solution of the form,

$$f = f_{amp}\, e^{-i(\omega t - \vec{k}\cdot\vec{r})}$$

$$\vec{\nabla} \times \vec{E} \xrightarrow{FFT} i\vec{K} \times \vec{E} \rightarrow i E_{\perp}$$

$$\frac{\partial}{\partial t}\vec{B} \xrightarrow{FFT} -i\omega\vec{B}$$

using Faraday's law,

$$iKE_{\perp} = i\omega B$$

$$\Rightarrow \frac{\omega}{K} = \left|\frac{E_{\perp}}{B}\right|$$

Characteristic velocity. $(\tilde{V})$

$$\Big(\frac{1}{c^2}\underbrace{\frac{\tilde{E}_{\perp}}{\tilde{B}}\frac{L}{T}}_{\tilde{V}^2}\Big) \rightarrow \frac{1}{c^2}\cdot\tilde{V}^2$$

$\tilde{V}^2 << c^2$

$\downarrow$

slow phenomena

negligible $\longrightarrow$ zero displacement current.

So the final form,

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$

CHALLENGE $\rightarrow$ COMBINE $\rightarrow$ $\vec{J}, \vec{E}, \vec{B}, \vec{u}$ $\rightarrow \rho$

(5) OHM's LAW:

External current

Neglecting the current due to moving net charge

$$\vec{J} = \underline{\sigma}(\vec{E} + \vec{u}\times\vec{B})$$

convective current

MOVING FRAME OF REFERENCE

Only comes from ABSOLUTE FRAME OF REFERENCE

Due to the assumption of ideal conductor

$\underline{\sigma} \to \infty$

In ideal MHD as $\underline{\sigma} \to \infty$, $\boxed{\vec{E} = -\vec{u}\times\vec{B}}$ For ideal MHD limit

is induced by the plasma moving across the magnetic field line.

Hence, within ideal MHD, electric fields cause no particle accelaration along magnetic field lines.

Since, all the fluctuations in the E-fields are in $\perp B$ direction, the velocity of a local fluid element can be expressed as

$$\vec{u} = \frac{\vec{E}\times\vec{B}}{B^2} \longrightarrow \vec{E} = 0$$

In the rest frame of plasma.

(6) EQN OF STATE:

Incompressible: $\vec{\nabla}\cdot\vec{u} = 0$

$$p = f(\rho, T)$$

RECALLING MOMENTUM EQN.

$$\frac{\partial}{\partial t}\vec{u} + \vec{u}\cdot\vec{\nabla}u = -\frac{\vec{\nabla}p}{\rho}$$

Neutral Fluid

OPERATING $\vec{\nabla}\cdot$

$$\vec{\nabla}\cdot\left(\frac{\partial}{\partial t}\vec{u}\right) + \vec{\nabla}\cdot(\vec{u}\cdot\vec{\nabla}\vec{u}) = -\vec{\nabla}\cdot\frac{\vec{\nabla}p}{\rho}$$

$$\Rightarrow \vec{\nabla}\cdot\left(\frac{1}{\rho}\vec{\nabla}p\right) = -\vec{\nabla}\cdot(\vec{u}\cdot\vec{\nabla}\vec{u}) - \frac{\partial}{\partial t}\cancel{(\vec{\nabla}\cdot\vec{u})}^{\,0}$$

For Compressible:

$$\rho = \rho_0\left(\frac{p}{p_0}\right)^{1/\gamma}$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\left(\frac{1}{\rho}\vec{\nabla}p\right) = -\vec{\nabla}\cdot(\vec{u}\cdot\vec{\nabla}\vec{u})}$$

Neutral Fluid

MHD

$$\boxed{\Rightarrow \vec{\nabla}\cdot\left(\frac{1}{\rho}\nabla p\right) = -\nabla\cdot(\vec{u}\cdot\vec{\nabla}\vec{u}) + \nabla\cdot(\vec{J}\times\vec{B})}$$

The equation of states for MHD defines the pressure independent of the history of the system for incompressible fluids.

no $\frac{\partial}{\partial t}$

For IDEAL MHD: (FLUID MECHANICS + PRE MAXWELL ELECTROMAGNETISM)

$$\frac{\partial}{\partial t}\vec{B} = \vec{\nabla}\times(\vec{u}\times\vec{B})$$

(FARADAY'S LAW)

CONSERVATION OF MAGNETIC FLUX

$$\frac{\partial}{\partial t}\rho + \vec{u}\cdot\nabla\rho = 0$$

$$\vec{\nabla}\cdot\vec{u} = 0$$

CONTINUITY EQN (incompressible)

CONSERVATION OF MASS

$$\rho\left(\frac{\partial}{\partial t}\vec{u} + \vec{u}\cdot\vec{\nabla}\,\vec{u}\right) = -\nabla p + \frac{1}{\mu_0}\left[\vec{\nabla}\times\vec{B}\right]\times\vec{B}$$

AMPERE'S LAW

CONSERVATION OF MOMENTUM

For compressible, $p = f(\rho)$ and need to modify continuity equ.

INTERESTING PHENOMENA IN MHD

→ unstable motions

→ waves

- → sound waves → Vibration of the fluids
- → Alfen waves → " " " magnetic field in presence of the fluid.
- → magnetosonic waves
  → coupling between fluid and the magnetic fields.