IDEAL MHD ($\sigma \to \infty$)

$$\frac{\partial}{\partial t}\vec{B} = \vec{\nabla} \times (\vec{u} \times \vec{B})$$

(FARADAY'S LAW)

CONSERVATION OF MAGNETIC FLUX

$$\frac{\partial}{\partial t}\rho + \vec{u} \cdot \nabla \rho = 0$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

CONTINUITY EQN (incompressible)

CONSERVATION OF MASS

$$\rho\left(\frac{\partial}{\partial t}\vec{u} + \vec{u} \cdot \vec{\nabla}\,\vec{u}\right) = -\nabla p + \frac{1}{\mu_0}\left[\vec{\nabla} \times \vec{B}\right] \times \vec{B}$$

AMPERE'S LAW

CONSERVATION OF MOMENTUM

For ideal MHD, conductivity $\sigma \to \infty$

$$\vec{J} = \sigma\,(\vec{E} + \vec{u} \times \vec{B})$$ (OHM'S LAW)

$$\vec{E} = -\vec{u} \times \vec{B}$$

All the electric fields are results of $\vec{u} \times \vec{B}$ drift

At this point, since every quantity is conserved, for ideal MHD every quantity can be traced back in time i.e. reversible.

However, if we add some viscosity or some dissipation it would result in loss or gain in energy in the system. Such MHD system are irreversible.

Another thing to notice in ideal MHD collisions do not enter into the force equ (momentum equ.)

Considering collision and momentum exchange would give rise to finite conductivity in plasma removing all the approximations we assigned for ideal MHD. Such theories are known as DISSIPATIVE MHD.

DISSIPATIVE MHD

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B} \qquad \vec{J} = \sigma(\vec{E}+\vec{u}\times\vec{B})$$

(AMPERE'S LAW) (FARADAY'S LAW) (OHM'S LAW)

Taking rotation over

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times(\mu_0\vec{J})$$

$$= \vec{\nabla}\times\mu_0\sigma(\vec{E}+\vec{u}\times\vec{B})$$

using Maxwell's equ.

$$\boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\cdot(\cancel{\vec{\nabla}\cdot\vec{B}}) - \nabla^2\vec{B} = -\nabla^2\vec{B}}$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \mu_0\sigma\left[\vec{\nabla}\times\vec{E} + \vec{\nabla}\times(\vec{u}\times\vec{B})\right]$$

$$\Rightarrow -\nabla^2\vec{B} = \mu_0\sigma\left[\vec{\nabla}\times\vec{E} + \vec{\nabla}\times(\vec{u}\times\vec{B})\right]$$

$$\Rightarrow \vec{\nabla}\times\vec{E} = -\frac{1}{\mu_0\sigma}\nabla^2\vec{B} - \vec{\nabla}\times(\vec{u}\times\vec{B})$$

Now, using FARADAY's LAW,

$$\Rightarrow \quad -\frac{\partial}{\partial t}\vec{B} = -\frac{1}{\mu_0 \sigma}\nabla^2\vec{B} - \vec{\nabla}\times(\vec{u}\times\vec{B})$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\vec{B} = \underbrace{\vec{\nabla}\times(\vec{u}\times\vec{B})}_{\text{Convective term}} + \underbrace{\frac{1}{\mu_0 \sigma}\nabla^2\vec{B}}_{\text{DIFFUSION}}$$

FARADAY'S LAW FOR DISSIPATIVE MHD

GENERAL DIFFUSION EQN: $\frac{\partial}{\partial t} f = D\nabla^2 f$

It determines how fast a field 'f' will spread depending on spatial gradient and diffusive const.

In general diffusion equ. is valid for scalar field. But in this with a mere comparison we can see the second term on the right hand side along with the standard convective term (1st term on RHS) forms a diffusion like equ. for vectorial magnetic field.

$\frac{1}{\mu_0 \sigma}\nabla^2\vec{B}$ → This term becomes important when magnetic Reynold's number becomes small.

Reynold's number

$$R_L = \mu_0 \sigma L u$$

conductivity

characteristic velocity

characteristic length scale

In summary, for small R_L, system appears to have finite conductivity which makes it dissipative.

Finally, for RESISTIVE MHD

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\frac{\partial}{\partial t} \vec{B} = \vec{\nabla} \times (\vec{u} \times \vec{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \vec{B}$$

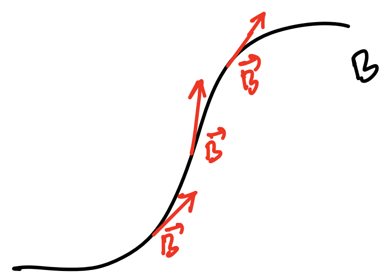
$$\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

$$\rho \left(\frac{\partial}{\partial t} \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} \right) = -\vec{\nabla} p + \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) \times \vec{B} + \cdots$$

$$p = f(\rho)$$

FROZEN IN FIELD LINES

In the limit of infinite conductivity $\sigma \to \infty$, B-field lines can be identified with particles.

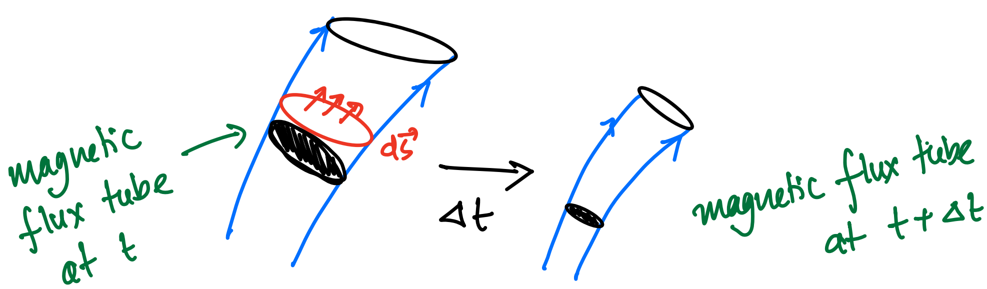


B field at any given point is the tangent on the field line.

Magnetic field lines at any given time can be defined as,

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z}$$

ALLOWING COMPRESSIBILITY



Let's introduce magnetic flux as $\phi(S_{t+\Delta t}, B_t)$

$$\phi\left(S_{t+\Delta t}, B_t\right) = \int_{S_{t+\Delta t}} \vec{B}_t \cdot d\vec{s}$$

normal to surface S

Using the definition above

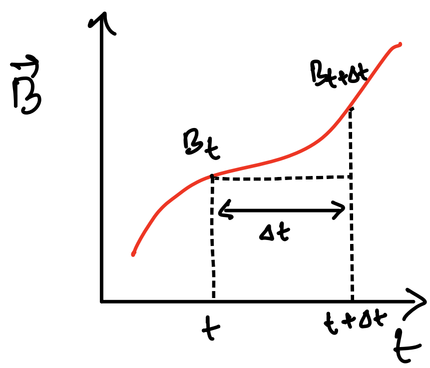
$$\phi\left(S_{t+\Delta t}, B_{t+\Delta t}\right) = \int_{S_{t+\Delta t}} \vec{B}_{t+\Delta t} \cdot d\vec{s} \qquad (1)$$

Assuming the change in B is slow we can do a series expansion of $\vec{B}_t$

$$\approx \Delta t \int_{S_{t+\Delta t}} \frac{\partial}{\partial t}\vec{B}_t \cdot d\vec{s} + \int_{S_{t+\Delta t}} \vec{B}_t \cdot d\vec{s} \qquad (2)$$

(taking only first two term)

NOTE: The above approximation is taken from finite difference approximation.



$$B_{t+\Delta t} = B_t + \frac{\partial B_t}{\partial t}\Delta t$$

We assume that Δt is small, which gives

$$\int_{S_{t+\Delta t}} \vec{B}_t \cdot d\vec{s} - \int_{S_t} \vec{B}_t \cdot d\vec{s} = O(\Delta t)$$

order of Δt

Now, taking time derivative over the magnetic field

$$\int_{S_{t+\Delta t}} \frac{\partial}{\partial t}\vec{B}_t \cdot d\vec{s} \approx \int_{S_t} \frac{\partial}{\partial t}\vec{B}_t \, d\vec{s} + O(\Delta t)$$

Using this back in equ. (2)

$$\approx \Delta t \int_{S_t} \frac{\partial}{\partial t}\vec{B}_t \cdot d\vec{s} + \int_{S_{t+\Delta t}} \vec{B}_t \cdot d\vec{s}$$

This is equivalent to the definition of our flux

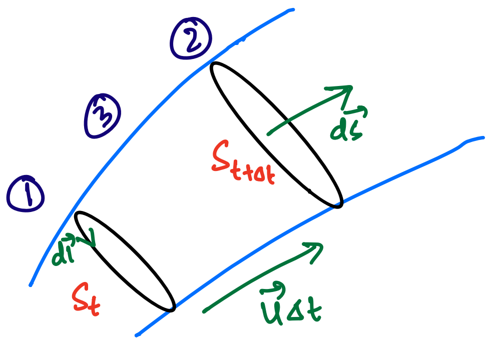
Now, using the defuniton and equ (1) we can write

$$\phi(S_{t+\Delta t}, B_t) = \phi(S_{t+\Delta t}, B_{t+\Delta t}) + \Delta t \int_{S_t} \frac{\partial}{\partial t}\vec{B}_t \cdot d\vec{s} \qquad (3)$$

Now, Since $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow$ NET FLUX $= 0$

Flux through the sides

$$\underbrace{\phi(S_t, B_t)}_{\substack{\text{START} \\ \text{Surface (1)}}} = \underbrace{\phi(S_{t+\Delta t}, B_t)}_{\substack{\text{END} \\ \text{Surface (2)}}} + \underbrace{\phi(S_{sw}, B_t)}_{\substack{\text{SIDES} \\ \text{Surface (3)}}} \qquad (4)$$



$d\vec{s}$ = surface normal

$|u\Delta t| \longrightarrow$ local distance between S_t and $S_{t+\Delta t}$

vector normal to the surface (sw): $\vec{a} = d\vec{l} \times \vec{u}\Delta t$

$$\phi(S_{sw}, B_t) = \oint B_t \cdot d\vec{l} \times \vec{u}\Delta t = \Delta t \oint (\vec{u} \times \vec{B}_t) \cdot d\vec{l}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b})$$

along the boundary of S_t

Now, we can use the Gauss law / Stokes theorem

$$\boxed{\phi(S_{sw}, B_t) = \Delta t \int_{S_t} \vec{\nabla} \times (\vec{u} \times \vec{B}_t)\, d\vec{s}} \quad (5)$$

The change in the flux,

$$\frac{\phi(S_{t+\Delta t}, B_{t+\Delta t}) - \phi(S_t, B_t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{d}{dt} \phi(S, B)$$

$$= \frac{1}{\cancel{\Delta t}}\Big[\underbrace{\cancel{\Delta t}\int_{S_t}\frac{\partial}{\partial t}\vec{B}_t\, d\vec{s} + \cancel{\phi(S_{t+\Delta t}, B_t)}}_{\text{From equ. (3)}} \underbrace{- \cancel{\phi(S_{t+\Delta t}, B_t)} - \cancel{\Delta t}\int_{S_t}\nabla\times(\vec{u}\times\vec{B}_t)\, d\vec{s}}_{\text{using equ. (4) and (5)}}\Big]$$

This is the temporal derivative of the magnetic flux when the field is changing with the plasma flow.

Finally, dropping the subscript t

$$\frac{d}{dt}\phi(S,B) = \int_S \overbrace{\left[\frac{\partial}{\partial t}\vec{B} - \vec{\nabla}\times(\vec{u}\times\vec{B})\right]}^{\to\, 0 \text{ for ideal MHD}} d\vec{s}$$

$\forall\, t$

$B = B(t)$

$S = S(t)$

Now, if we compare this with ideal MHD, the bracketed term on the right hand size becomes zero for ideal MHD.

For ideal MHD, the conductivity become infinite which allows the magnetic field at each point to vary in such a way that it's flux through any material surface (which is determined by the plasma) that is following the fluid is constant. Therefore it enables to attach the magnetic flux to the particles.

Naturally, we do not consider the movement of magnetic field lines as they are abstract concept. But in case of IDEAL MHD as they are frozen, the movement of the lines can be visualized by following the particles. The particles will act as marker for such case.

So, for the DISSIPATIVE MHD, $\frac{d}{dt}\phi(S,B)$ is not zero as,

$$\boxed{\frac{\partial}{\partial t}\vec{B} - \vec{\nabla}\times(\vec{u}\times\vec{B}) = \frac{1}{\mu_0 \underline{\sigma}}\nabla^2\vec{B}}$$

Finite conductivity (resistivity) allows particles to be disconnected from the field lines. For example, magnetic reconnection.