MAGNETIC PRESSURE

MOMENTUM EQN:

$$\rho\left(\frac{\partial}{\partial t}\vec{u} + \vec{u}\cdot\vec{\nabla}\,\vec{u}\right) = -\nabla p + \underbrace{\boxed{\frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}}}_{\vec{J}\times\vec{B}} + \cdots$$

Reorganizing the second term on the R.H.S. using vector algebra

$$\vec{B}\times(\vec{\nabla}\times\vec{B}) = (\vec{\nabla}\cdot\vec{B})\,\vec{B} - (\vec{B}\cdot\vec{\nabla})\vec{B}$$

$$= \frac{1}{2}\vec{\nabla}(\vec{B}\cdot\vec{B}) - (\vec{B}\cdot\vec{\nabla})\vec{B}$$

$$= \frac{1}{2}\vec{\nabla}B^2 - (\vec{B}\cdot\vec{\nabla})\vec{B}$$

Now, let's rewrite the momentum equ.

$$\rho\left(\frac{\partial}{\partial t}\vec{u} + \vec{u}\cdot\vec{\nabla}\,\vec{u}\right) = \boxed{-\nabla\left(p + \frac{1}{2\mu_0}B^2\right)} + \frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B} + \cdots$$

Magnetic field in the same mathematical form of plasma pressure.

In the context of MHD, in analogy to particle pressure the strength of the magnetic field will act in a similar way.

In general form,

$$\boxed{\vec{J}\times\vec{B} = \frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B} = \nabla\cdot\Pi^M}$$

Magnetic tensor

$$\Pi^M = \frac{1}{\mu_0}\begin{bmatrix} B_x^2 - \frac{1}{2}B^2 & B_xB_y & B_xB_z \\ B_yB_x & B_y^2 - \frac{1}{2}B^2 & B_yB_z \\ B_zB_x & B_zB_y & B_z^2 - \frac{1}{2}B^2 \end{bmatrix}$$

We can simplify the form of magnetic tensor (π^M) by considering the following.

i) The magnetic field lines are straight and parallel

ii) The intensity of the magnetic field changes in $\perp$ direction to $\vec{B}$.

This allows us to make $(\vec{B}\cdot\vec{\nabla})\vec{B} = 0$ which implies the total momentum comes from the pressure term.

$$\vec{J}\times\vec{B} = -\vec{\nabla}\left(p + \frac{1}{2\mu_0}B^2\right)$$

Now, additionally if we consider the magnetic field is $\parallel \hat{z}$,

$$\pi^M = \frac{1}{2\mu_0}\begin{bmatrix} -B^2 & 0 & 0 \\ 0 & -B^2 & 0 \\ 0 & 0 & B^2 \end{bmatrix}$$

$\perp B$, magnetic presure

$\parallel B$, tension

Rewriting magnetic tensor in its general form,

$$\pi^M = \frac{1}{\mu_0}\begin{bmatrix} -\frac{1}{2}B^2 & 0 & 0 \\ 0 & -\frac{1}{2}B^2 & 0 \\ 0 & 0 & B_z^2-\frac{1}{2}B^2 \end{bmatrix}$$

$$\pi^M = \frac{1}{\mu_0}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_z^2 \end{bmatrix} - \frac{1}{2\mu_0}B^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

TENSION along $\hat{z}$

Isotropic pressure of magnetic field

Normal pressure, $\nabla p = \nabla(p\mathbb{1}) = \nabla\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$

Similar representation

LOCAL PRESSURE

$\frac{B^2}{2\mu_0}$ is $\perp$ to B-field lines

TENSION

$\frac{B^2}{2\mu_0}$ is $\parallel$ to B-field lines

PLASMA β

Plasma β is the relative importance of plasma pressure with respect to magnetic pressure.

$$\boxed{\beta = \frac{nkT}{B^2/2\mu_0}}$$

← particle pressure / plasma pressure

← magnetic pressure

NOTE: β could be related to characteristic velocities as

$$\beta = \frac{C_s^2}{V_A^2} \quad \begin{array}{l} \leftarrow \text{sound speed} \\ \leftarrow \text{magnetic field (Alfen)} \end{array}$$

In. solar plasma $\quad \beta = \{0.5, 1\}$

Good interplay between particle pressure and magnetic pressure

In laboratory $\quad \beta = \sim 10^{-8}$

Magnetic pressure largly dominates in controlled environment.

STATIONARY SOLUTION

Assuming $\vec{u} = 0$

The L.H.S. of our momentum equ. becomes zero. So, we have,

$$\nabla\left(p + \frac{1}{2\mu_0} B^2\right) = \frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B}$$

Now. assume $\vec{B} \parallel \hat{z}$ and only variation is in $\hat{x}$

$\Rightarrow B = [0, 0, B_z(x)], \quad p = p(x)$ Same for the pressure

$$\nabla\left(p + \frac{1}{2\mu_0} B^2\right) = \frac{d}{dx}\left(p + \frac{1}{2\mu_0} B^2\right) = 0$$

$$\Rightarrow \quad p + \frac{1}{2\mu_0} B^2 = \text{Const.}$$ $\longrightarrow$ Plasma is DIAMAGNETIC

i.e. if $p \uparrow$, then $B \downarrow$

Consider a large no. of particles and each of them are gyrating. Total contribution due to gyration will impact the external magnetic field. Increasing the no. of particle will reduce the magnetic pressure.

PLASMA PINCHES

It was introduced to confine plasma to make it sustainable.

IMPORTANT FEATURES!

- They are generally studied in cylindrical coordinate.
- No radial component of B, $B_r = 0$
- p, B do not depend on $\hat{z}, \hat{\theta}$

So, $\vec{B}$ is expressed as,

$$\vec{B} = B_\theta(\vec{r})\hat{\theta} + B_z(\vec{r})\hat{z}$$

Using Ampere's law : $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

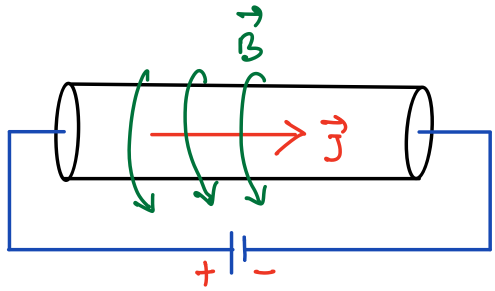
General Current expression

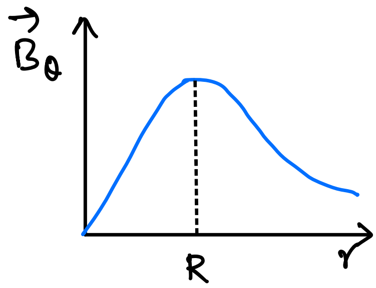
$$\vec{J} = \frac{1}{\mu_0}\left\{0, -\frac{dB_z}{dr}, \frac{1}{r}\frac{d(rB_\theta)}{dr}\right\}$$

Z-pinch

$$B_z = 0 \,, \quad B = \{0, B_\theta, 0\}$$

Current only in z direction.





θ-pinch

$$B = \{0, 0, B_z(r)\}$$

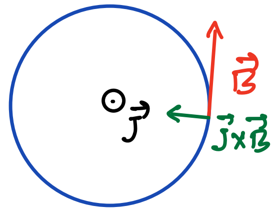
This will have purely azimuthal currents. (θ direction)

SCREW pinch

Combination of θ pinch + Z pinch

PINCH INSTABILITIES

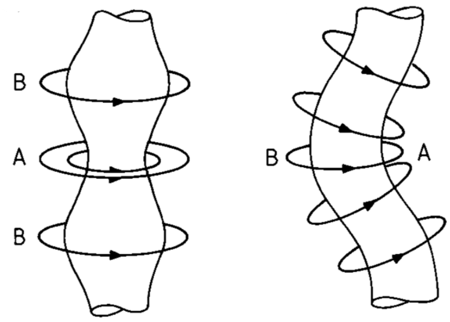
Considering different pinch cases, a small perturbation in current could lead to exponentially large variation making the system unstable.



The pinch mechanism was supposed to confine the plasma using $\vec{J}\times\vec{B}$ force.

Two examples of different perturbations leading to instabities.

Mag. pressure
$A > B$



Mag. pressure
$A > B$

SASUAGE INSTABILITY

KINK INSTABILITY

APPLICATION OF PINCH IN FUSION