DYNAMIC SOLUTIONS TO MHD EQUATIONS

An introduction to ALFÉN waves

INCOMPRESSIBLE IDEAL MHD:

$\vec{\nabla}\cdot\vec{u} = 0$ $\quad \beta \to \infty$

Our incompressible MHD equs.

$$\frac{\partial}{\partial t}\vec{B} = \vec{\nabla}\times(\vec{u}\times\vec{B}) \qquad \text{(FARADAY'S LAW)}$$

$$\frac{\partial}{\partial t}\rho + \vec{u}\,\nabla\rho = 0 \qquad \text{(CONTINUITY EQN)}$$

$$\vec{\nabla}\cdot\vec{u} = 0$$

$$\rho\left(\frac{\partial}{\partial t}\vec{u} + \vec{u}\cdot\vec{\nabla}\,\vec{u}\right) = -\vec{\nabla}p + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B}$$

(MOMENTUM EQN)

(AMPERE'S LAW)

Here, we are going to use the linearization technique by introducing small perturbation.

1. Let's assume our steady state solutions are in the following form,

$$\boxed{\vec{B} = B_0\hat{z}\,,\quad \rho = \rho_0\,,\quad p = p_0\,,\quad \vec{u} = 0}$$

② PERTURBATION:

$$\vec{B} = \vec{B}_0 + \vec{B}_1 \,,\quad \vec{u} = 0 + \vec{u}_1 \,,\quad \rho = \rho_0 + \rho_1 \,,\quad p = p_0 + p_1$$

$\vec{B}_0 \gg \vec{B}_1$ and similar for all the quantities.

③ LINEARIZATION: (only linear terms will survive)

Now, substituting the perturbed quantities in the main set of equs. and ignoring higher order terms (Similar to the DYNAMIC SOLUTIONS we learned earlier)

After linearization,

$$\frac{\partial}{\partial t}\vec{B}_1 = \vec{\nabla}\times(\vec{u}_1\times\vec{B}_0)$$

$$\frac{\partial}{\partial t}\rho_1 = 0$$

$$\vec{\nabla}\cdot\vec{u}_1 = 0$$

$$\rho_0\frac{\partial}{\partial t}\vec{u}_1 = -\nabla p_1 + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}_1)\times\vec{B}_0$$

④ Assumption: The plasma supports wave like motion and have a form $\exp[-i(\omega t - \vec{k}\cdot\vec{r})]$

$$\begin{pmatrix}\vec{B}_1\\ \vec{u}_1\\ \vec{\rho}_1\\ \vec{p}_1\end{pmatrix} = \begin{pmatrix}\tilde{\vec{B}}_1\\ \tilde{\vec{u}}_1\\ \tilde{\rho}_1\\ \tilde{p}_1\end{pmatrix} e^{-i(\omega t - \vec{k}\cdot\vec{r})}$$

It will allow us to rewrite the time and spatial derivatives in spectral form,

$$\boxed{\frac{\partial}{\partial t} \to -i\omega \qquad \vec{\nabla} \to i\vec{k}}$$

(5) Using the above expressions, we can rewrite our linearized equations,

using BAC−CAB

$$-i\omega\vec{B}_1 = i\vec{k}\times(\vec{u}_1\times\vec{B}_0) = i(\vec{k}\cdot\vec{B}_0)\vec{u}_1 - i(\vec{k}\cdot\vec{u}_1)\vec{B}_0 \rightarrow (1)$$

$$-i\omega\rho_1 = 0 \rightarrow (2)$$

$$i\vec{k}\cdot\vec{u}_1 = 0 \rightarrow (3)$$

$$-i\rho_0\omega\vec{u}_1 = -i\vec{k}p_1 + \frac{i}{\mu_0}(\vec{k}\times\vec{B}_1)\times\vec{B}_0 \rightarrow (4)$$

Assuming $\vec{k}$ is real and ω complex, we can write from the incompressibility condition,

(3) $\rightarrow$ $\boxed{\vec{k}\perp\vec{u}_1}$ i.e. the wave vector is $\perp$ to the fluctuations in velocity. (TRANSVERSE)

Our continuity equ. gives us

(2) $\rightarrow$ $\boxed{\rho_1 = 0}$ which also make sense because of our incompressibility.

LITTLE BACKGROUND

we had, $\frac{\partial}{\partial t}\rho_1 = 0 \Rightarrow \rho_1 = \text{const.}$

Now, $\omega \neq 0$, to satisfy equ. (2) only logical choice for $\rho_1 = 0$

Now, in equ. 1 , the second term on R.H.S.

1 $\Rightarrow$ $-i\omega \vec{B}_1 = i(\vec{K}\cdot\vec{B}_0)\vec{u}_1 - i(\vec{K}\cdot\vec{u}_1)\vec{B}_0$

$K \perp u_1$

$$\Rightarrow -i\omega\vec{B}_1 = i(\vec{K}\cdot\vec{B}_0)\vec{u}_1 \rightarrow 5$$

implies perturbations in $\vec{B}$ are parallel to perturbations in $\vec{u}$

$$\frac{B_1}{u_1} = -\vec{K}\cdot\vec{B}_0 \cdot \frac{1}{\omega}$$

The magnetic field will oscillate in a proportion to the velocity vector in a direction $\perp$ to $\vec{B}_0$

Background mag. field

4 $\Rightarrow$ Let's take the second term on RHS

$$-\left[\vec{B}_0 \times (\vec{K}\times\vec{B}_1)\right] = -\left[\vec{K}(\vec{B}_0\cdot\vec{B}_1) - \vec{B}_1(\vec{B}_0\cdot\vec{K})\right]$$

putting it back in 4 and multiplying i on both sides

$$\rho_0\,\omega\vec{u}_1 = \vec{K}p_1 + \frac{1}{\mu_0}\left[\vec{K}(\vec{B}_0\cdot\vec{B}_1) - \vec{B}_1(\vec{B}_0\cdot\vec{K})\right] \rightarrow 6$$

Now, using 5

$$\vec{B}_1 = -\frac{\vec{K}\cdot\vec{B}_0}{\omega}\vec{u}_1 \quad \text{implies, } \vec{B}_1 \parallel \vec{u}_1$$

Again 6 $\Rightarrow$

$$\omega\rho_0\vec{u}_1 = \vec{K}p_1 + \frac{\vec{K}}{\mu_0}(\vec{B}_0\cdot\vec{B}_1) + \frac{1}{\mu_0}\frac{(\vec{B}_0\cdot\vec{K})^2}{\omega}\vec{u}_1$$

REARRANGING

$$\left[\omega\rho_0 - \frac{1}{\mu_0\omega}\left(\vec{k}\cdot\vec{B}_0\right)^2\right]\vec{u}_1 = \left[p_1 + \frac{1}{\mu_0}\vec{B}_0\cdot\vec{B}_1\right]\vec{k} \qquad (7)$$

Now, we already chose $\vec{k}$ to be real and we have the same freedom to choose $\vec{u}_1$ to be real. Hence, the only way the above equ. (7) is valid, if the individual coefficients are zero.

$$\omega\rho_0 - \frac{1}{\mu_0\omega}\left(\vec{k}\cdot\vec{B}_0\right)^2 = 0$$

$$\Rightarrow \boxed{\omega^2 = \frac{1}{\mu_0\rho_0}\left(\vec{k}\cdot\vec{B}_0\right)^2}$$

DISPERSION RELATION FOR ALFVÉN WAVES

Consequence of FROZEN-IN-FIELD Lines

$$\omega^2 = \frac{1}{\mu_0\rho_0}k^2\left(\hat{k}\cdot\vec{B}_0\right)^2$$

$$\Rightarrow \left(\frac{\omega}{k}\right)^2 = \frac{1}{\mu_0\rho_0}\left(\hat{k}\cdot\vec{B}_0\right)^2$$

$$\Rightarrow \frac{\omega}{k} = \pm\frac{1}{\mu_0\rho_0}B_0\cos\theta$$

θ is the angle between $\vec{B}_0$ and $\vec{k}$

ALFVE'N VELOCITY

$$V_A = \frac{B_0}{\mu_0\rho_0}$$

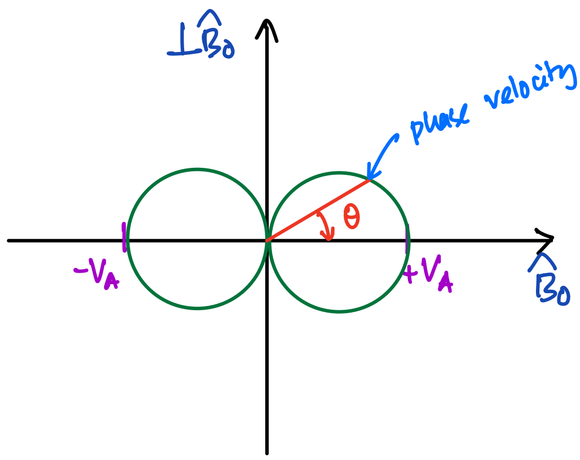
$$\Rightarrow \boxed{\frac{\omega}{k} = \pm V_A\cos\theta}$$

phase velocity

We also have,

$$\vec{u}_1 \perp \vec{k} \quad \text{(from incompressibility)}$$

implies, the waves are transverse and referred as SHEAR ALFVEŃ wave.

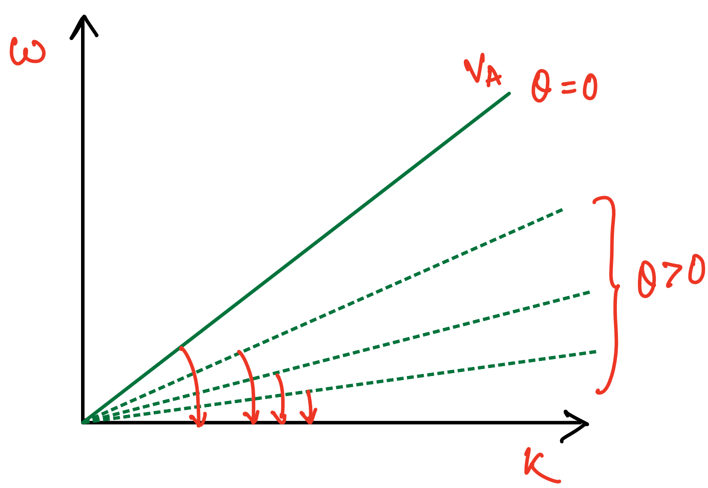


For a given θ

$\frac{\omega}{k}$ is const

For $\theta = 0°$

$\frac{\omega}{k} = V_A$ (maximum)



GROUP VELOCITY (ENERGY)

$$\frac{\partial \omega}{\partial K} = \nabla_K \omega$$

$$\omega^2 = V_A^2 K_z^2 \qquad \text{Considering, } \vec{B_0} = B_0 \hat{z}$$

$$\Rightarrow \omega = \pm V_A K_z$$

$$\boxed{\Rightarrow \nabla_K \omega = \pm V_A \hat{z}}$$

Important thing to notice, the group velocity does not depend on $\vec{K}$ instead it is always along the background magnetic field $\vec{B_0}$.

Therefore, the propagation of information and energy is always along the background magnetic field line.

Now, lets get back to equ. (7) Considering the coefficient on R.H.S.

$$\left[p_1 + \frac{1}{\mu_0} \vec{B_0} \vec{B_1} \right] = 0$$

$$\Rightarrow p_1 = -\frac{1}{\mu_0} \vec{B_0} \vec{B_1}$$

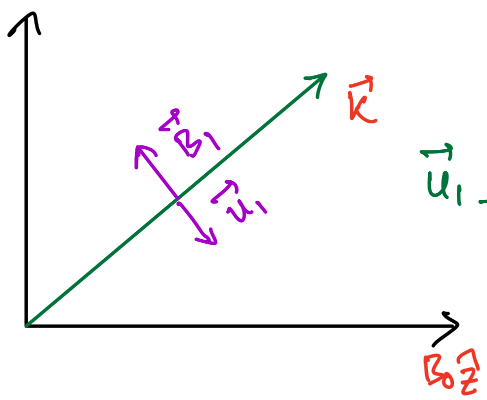
We also have, $\vec{B_1} \parallel \vec{u_1}$ which gives $\vec{B_1} = -\frac{\vec{K} \cdot \vec{B_0}}{\omega} \vec{u_1}$

$$\boxed{p_1 = \frac{1}{\mu_0 \omega} (\vec{B_0} \cdot \vec{K})(\vec{B_0} \cdot \vec{u_1})} \rightarrow (8)$$

This is equivalent to EQN. OF STATE for incompressible MHD.

We can write, $p_1 = p_1(\vec{B}_1)$

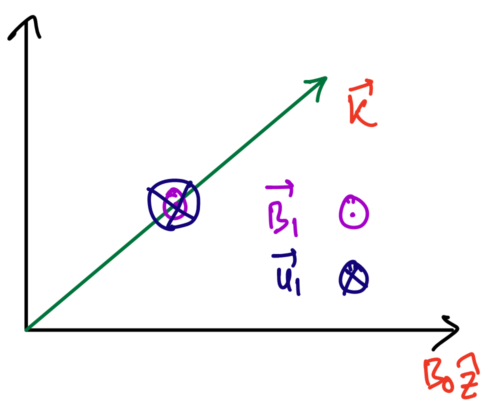
The pressure will depend on polarization of the waves.



$\theta = 0°$

if we have $\vec{k} \parallel \vec{B}_0$ and $\vec{u}_1 \perp \vec{k}$, then $\vec{u}_1 \perp \vec{B}_0$ - From Eqn. (8)

NO PRESSURE VARIATIONS

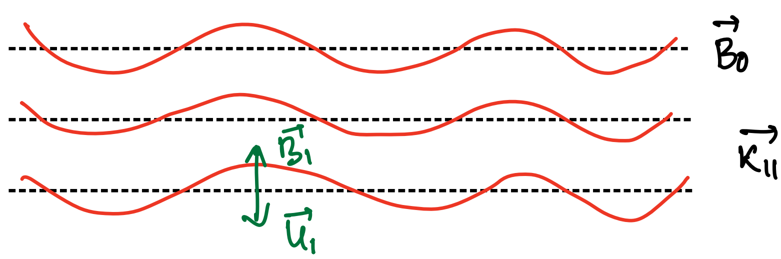


if we have $\vec{B}_1, \vec{u}_1 \perp$ plane $\vec{k}$ and $\vec{B}_0$

$p = 0$

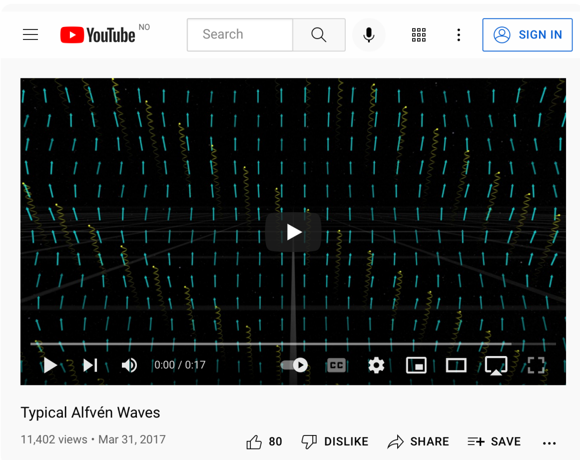
$\Rightarrow$ SHEAR ALFVÉN WAVES

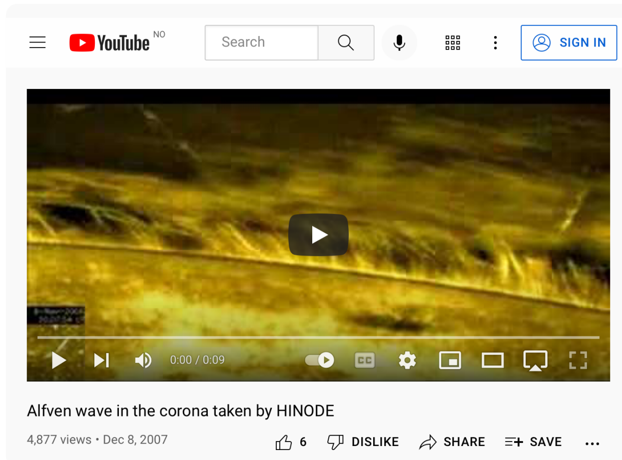
Example:



Remember, all the velocity is due to $\vec{E} \times \vec{B}$ in the plasma.

$$\boxed{V_1 = \frac{\vec{E}_1 \times \vec{B}_0}{B_0^2}}$$





ENERGY DENSITY FOR SHEAR ALFVÉN WAVE

Ratio of electric and magnetic field energy,

$$\boxed{\frac{W_E}{W_B} = \frac{\frac{1}{2}\epsilon_0 |E_1|^2}{\frac{1}{2}|B_1|^2/\mu_0} = \frac{|E_1|^2}{|B_1|^2}\frac{1}{c^2}}, \text{ where } c^2 = \frac{1}{\mu_0 \epsilon_0}$$

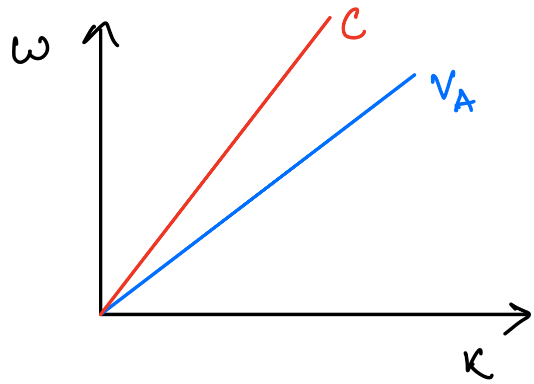
From, FARADAY'S LAW, $\boxed{i\vec{k} \times \vec{E}_1 = i\omega \vec{B}_1}$

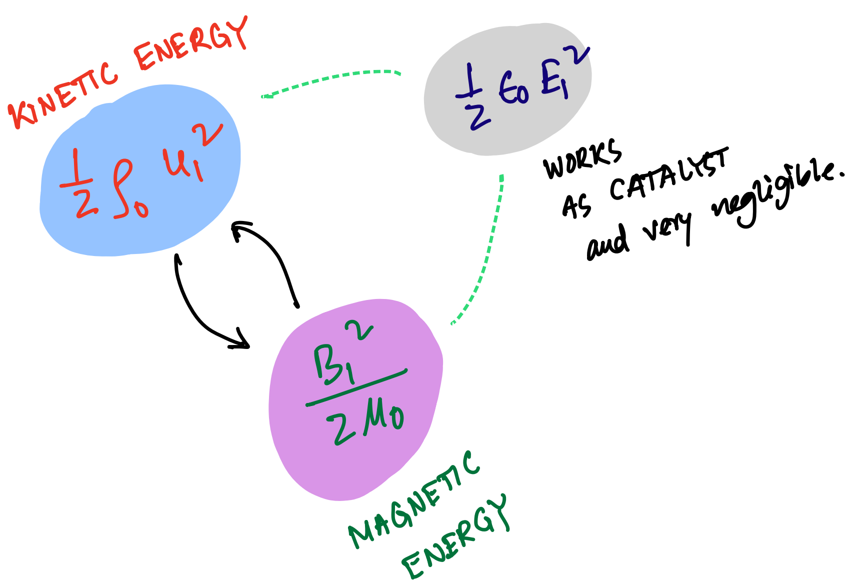
For SHEAR ALFVE'N wave,

$$\vec{K} \perp \vec{E_1} \Rightarrow \frac{|E_1|}{|B_1|} = \frac{\omega}{K} = V_A$$

$$\Rightarrow \frac{W_E}{W_B} = \frac{V_A^2}{C^2} \ll 1$$

Therefore, the electric energy is negligible in comparison to magnetic field energy.





PLASMA β and ALFVÉN WAVE

multiplying and dividing by M

$$\beta = \frac{nKT}{\frac{B^2}{2\mu_0}} = \frac{\rho KT}{M\, B^2/2\mu_0} = \frac{KT}{M} \cdot \frac{2\mu_0 \rho}{B^2}$$

$$\approx 2\frac{C_s^2}{V_A^2}$$

Compressible

$$\boxed{\Rightarrow \quad \beta \approx 2\frac{C_s^2}{V_A^2} \approx 2\frac{r_L^2\,\Omega_{pi}^2}{c^2}}$$

incompressible

The assumption of incompressibility also assumes that all velocities of propagation are well below sound speed. Which means, $\beta \gg 1$

But in practical $\beta \ll 1$ or $\beta \approx 1$. Therefore, we may need to think about V_A.