Compressional Alfvén Wave

Revisiting Alfve'n waves in incompressible ideal MHD

For incompressible ideal MHD:

$$\gamma \to \infty \quad , \quad \vec{\nabla}\cdot\vec{u} = 0$$

The dispersion relation:

$$\left[\omega\rho_0 - \frac{1}{\mu_0\omega}(\vec{k}\cdot\vec{B_0})^2\right]\vec{u_1} = \left[p_1 + \frac{1}{\mu_0}\vec{B_0}\cdot\vec{B_1}\right]\vec{k}$$

$$\Rightarrow \boxed{\omega^2 = \frac{1}{\mu_0\rho_0}(\vec{k}\cdot\vec{B_0})^2}$$

$$\Rightarrow \boxed{\frac{\omega}{k} = \pm V_A \cos\theta}$$

PHASE VELOCITY

θ is $\measuredangle\{\vec{B_0},\vec{k}\}$

$$\boxed{V_A = \frac{B_0}{\mu_0\rho_0}}$$

ALFVE'N VELOCITY

Wave is linearly polarized ($\vec{k}$ is real)

" " transverse (from compressibility, $\vec{k}$ is orthogonal to $\vec{u_1}$)

Allowing compressibility would give us,

$$\vec{\nabla}\cdot\vec{u} \neq 0$$

So, we need a equ. of state of the form

$$p = p(\rho)$$

For isothermal motion:

$$p = nKT = \rho \frac{KT}{M}$$

NOTE: We can also have adiabatic system.

LINEARIZED EQUATIONS

Note: Introduce the first order perturbed quantities to the compressible MHD equations and linearize.

1) $\frac{\partial}{\partial t}\vec{B}_1 = \vec{\nabla} \times (\vec{u}_1 \times \vec{B}_0)$

2) $\frac{\partial}{\partial t}\rho_1 + \vec{\nabla}\cdot(\vec{u}_1 \rho_0) = 0$

$\Rightarrow \frac{\partial}{\partial t}\rho_1 + \rho_0 \vec{\nabla}\cdot\vec{u}_1 = 0$

3) $\rho_0 \frac{\partial}{\partial t}\vec{u}_1 = -\vec{\nabla}p_1 + \frac{1}{\mu_0}[\vec{\nabla}\times\vec{B}_1]\times\vec{B}_0$

4) $p_1 = \frac{KT}{M}\rho_1$

Assuming PLANE WAVE solution

$$e^{-i(\omega t - \vec{k}\cdot\vec{r})}$$

$$\frac{\partial}{\partial t} \rightarrow -i\omega \qquad \vec{\nabla} \rightarrow i\vec{k}$$

Substituting operators,

using BAC − CAB

1) $-i\omega\vec{B}_1 = i\vec{k}\times(\vec{u}_1\times\vec{B}_0) = i(\vec{k}\cdot\vec{B}_0)\vec{u}_1 - i(\vec{k}\cdot\vec{u}_1)\vec{B}_0$

2) $-i\omega\rho_1 + i\rho_0\,\vec{k}\cdot\vec{u}_1 = 0$

Note: Remember in case of incompressible fluids, density is constant but not here.

3) $-i\omega\rho_0 \vec{u}_1 = -i\vec{k}p_1 + \frac{i}{\mu_0}(\vec{k}\times\vec{B}_1)\times\vec{B}_0$

$\Rightarrow -i\omega\rho_0 \vec{u}_1 = -i\vec{k}p_1 - \frac{i}{\mu_0}\left[(\vec{B}_0\cdot\vec{B}_1)\vec{k} - (\vec{k}\cdot\vec{B}_0)\vec{B}_1\right]$

$\Rightarrow \rho_0\omega\vec{u}_1 = \vec{k}p_1 + \frac{1}{\mu_0}\left[(\vec{B}_0\cdot\vec{B}_1)\vec{k} - (\vec{k}\cdot\vec{B}_0)\vec{B}_1\right]$

4) $p_1 = \frac{KT}{M}\rho_1$

1 $\Rightarrow$ The fluctuations in the magnetic field are orthogonal to wave propagation vector,

$\vec{B}_1 \perp \vec{k}$

! Same as incompressible fluids

2 $\Rightarrow$ Fluctuation in velocity may have component along $\vec{k}$

$\vec{k}\cdot\vec{u}_1 = \omega\rho_1/\rho_0$

it's not orthogonal any more $\vec{k} \not\perp \vec{u}_1$

instead it will depend on fluctuations in mass density.

Now, let's use these equations and find out the dispersion relations for different cases.

For limiting case, $T=0$

Using equ. of state,

$p \to 0$

$\rho_1 \to$ can be neglected

Then, we end up with dispersion relation,

$$\left(\frac{\omega^4}{k^4} - \frac{\omega^2}{k^2} V_A^2\right)\left(\frac{\omega^2}{k^2} - V_A^2 \cos^2\theta\right) = 0$$

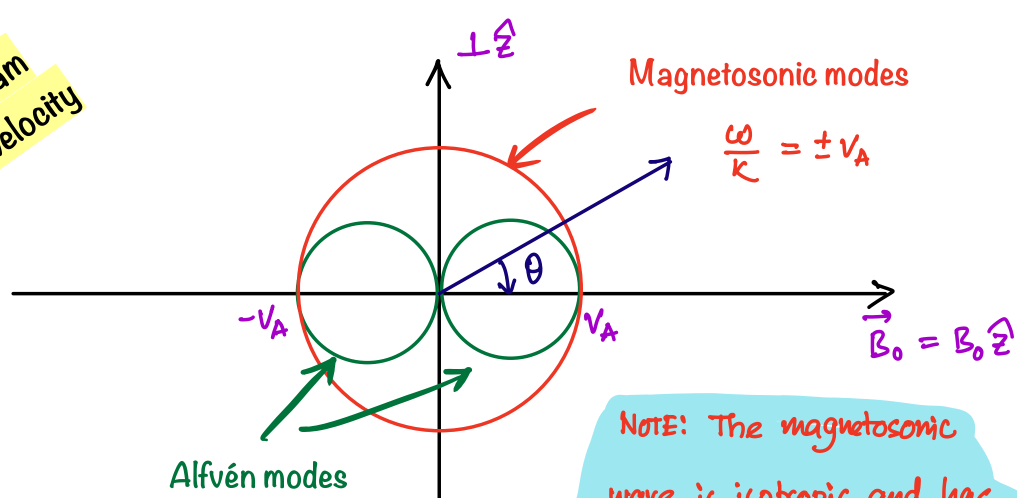
Sound wave   Alfvén wave

MAGNETOSONIC WAVE

Question: Why do we have sound wave even if we do not have pressure?

It comes from Magnetic pressure.

Polar diagram for phase velocity



NOTE: The magnetosonic wave is isotropic and has the same phase velocity with Alfvén

General case

We allow temperature (i.e. $T \neq 0$)

The modified dispersion equation will be the following,

$$\underbrace{\left(\frac{\omega^4}{k^4} - \frac{\omega^2}{k^2}\left(C_s^2 + V_A^2\right) + C_s^2 V_A^2 \cos^2\theta\right)}_{\text{MAGNETOSONIC WAVE}} \underbrace{\left(\frac{\omega^2}{k^2} - V_A^2 \cos^2\theta\right)}_{\text{Alfvén wave}} = 0$$

where, $C_s = \sqrt{\frac{KT}{M}}$

For Magnetosonic mode,

$$\frac{\omega^2}{k^2} = \frac{1}{2}\left(C_s^2 + V_A^2\right) \pm \frac{1}{2}\sqrt{\left(C_s^2 + V_A^2\right) - 4 C_s^2 V_A^2 \cos^2\theta}$$

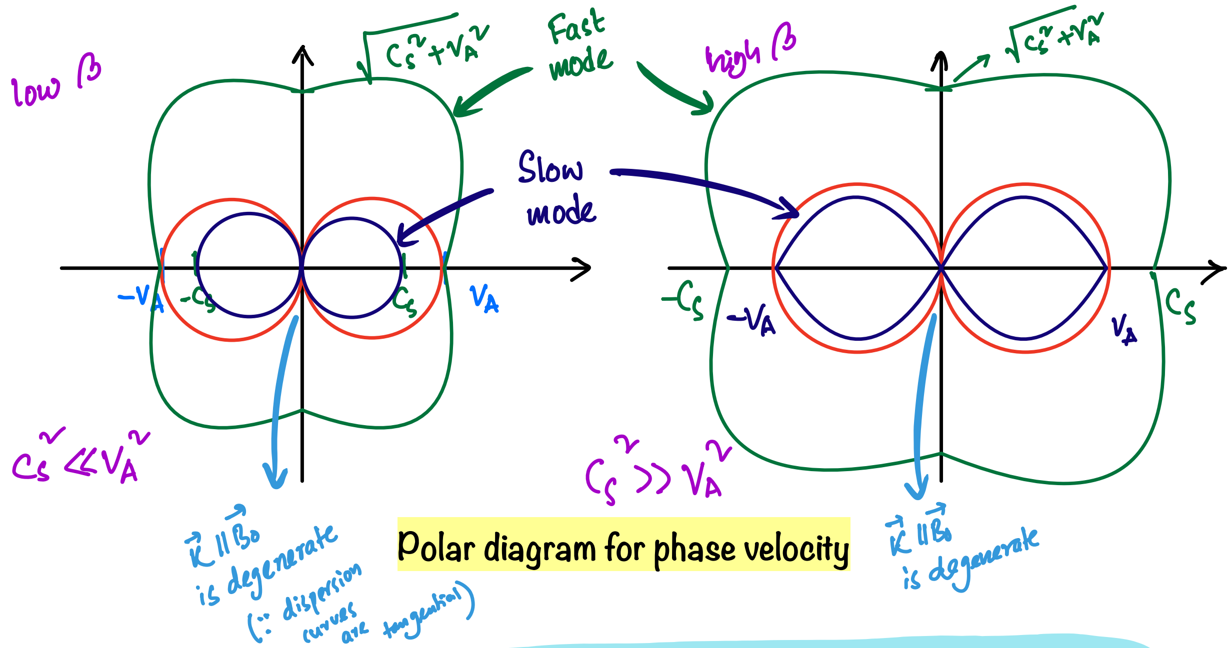
If the magnetosonic mode propagates along $\vec{B}_0$

i.e. $\theta = 0^\circ$

$$\underbrace{\frac{\omega^2}{k^2} = C_s^2}_{\text{SOUND WAVE}} \quad \text{and} \quad \underbrace{\frac{\omega^2}{k^2} = V_A^2}_{\text{AlFVE'N WAVE}}$$

If $\theta = 90^\circ$

$$\underbrace{\frac{\omega^2}{k^2} = C_s^2 + V_A^2}_{\text{FAST MODE}}$$



Polar diagram for phase velocity

If we compare three different polar diagrams for phase velocities, we can see the slow mode starts to shrink as we decrease C_s. For the cold case (i.e. $C_s = 0$) it shrinks to a point.

For low β, fast wave mode and Alfve'n wave merge ($\theta = 0°$) and for high β, slow wave mode and Alfve'n wave merge.

IMPORTANT: Sheared Alfve'n wave remains incompressible even when we allowed Compressibility.
(Can be derived taking a scalar product with $\vec{k}$ of ③)

If the angle of propagation is fixed, the phase velocity stays the same.

Magnetosonic wave will propagate ⊥ to the magnetic field.

All the waves can be damped with finite resistivity.