IDEAL ELECTRON MHD

In Ideal MHD, the general concept is to consider the plasma dynamics in the direction perpendicular to the local magnetic field and controlled by the bulk plasma velocity $\vec{E} \times \vec{B}/B^2$.

In this particular theory:
we assume,

1) Ions are immobile and act as neutralizing background

2) Electrons are moving

3) ignore pressure forces and resistivity → high conductivity

4) For frequencies:

$$\omega \ll \omega_{ce} \quad \text{but} \quad \omega \gg \Omega_{ci}$$

i.e. ions can't respond to any perturbation but the electrons can. Such a scenario will allow the electrons to have a velocity across B-field and will be given by $\frac{\vec{E} \times \vec{B}}{B^2}$

5) Electrons are considered light, hence the inertia can be neglected.

Under such assumptions electron dynamics can be given by INERTIA FREE DESCRIPTION

- Momentum Equ.

Since we are considering high frequency compared to classical MHD freq.

$$\rho\left(\cancel{\frac{\partial \vec{u}}{\partial t} + \vec{u}\,\vec{\nabla}\,\vec{u}}\right) = -en\vec{E} + \vec{J}\times\vec{B}$$

0

No pressure as we ignored any contribution.

$$\Rightarrow -en\vec{E} + \vec{J}\times\vec{B} = 0$$

$$\vec{J}\times\vec{B} \perp \vec{B}$$

$$\Rightarrow \vec{E} \perp \vec{B} \quad \text{at all times}$$

$$\Rightarrow -en\vec{E} + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})\times\vec{B} = 0$$

- Only source of $\vec{E}$ is,

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

FARADAY'S LAW

- Ampere's law :

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

Note: We are ignoring Maxwell's displacement current as we are still in MHD limit

Which gives us,

$$\vec{\nabla}\cdot\vec{J} = 0$$

$$\Rightarrow \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

flow velocity

charge density

Implicit Incompressibility

Our incompressible inertialess model allows us to ignore the electron continuity equ. Since we assumed ions are immobile, they have a uniform density. The initial electron density should have the same configuration and the basic equs will then assume it to be constant for all times. This appears due to the assumption that there is no effective space charge and the main source for electric field is the time varying magnetic field (Faraday's law).

This assumption is also valid for weakly inhomogeneous case.

The basic equs. for Electron MHD:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \rightarrow (1)$$

$$-en\vec{E} + \frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) \times \vec{B} = 0 \quad \rightarrow (2)$$

The dynamical wave form found in electron MHD is known as Whistler waves.

DYNAMIC SOLUTIONS:

ASSUMING

background density $n_0 \rightarrow$ uniform

background mag. field $B_0 \rightarrow$ homogeneous.

After LINEARIZATION of the eqns. ① and ② we get the following.

HINT: Take $\vec{\nabla}\times$ ② then substitute ① in ②

$$\vec{\nabla}\times\left[(\vec{\nabla}\times\vec{B}_1)\times\vec{B}_0\right] = -e\mu_0 n_0 \frac{\partial}{\partial t}\vec{B}_1$$

Using $\vec{\nabla}\times(\vec{A}\times\vec{B}) = \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A}) + (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B}$

$$\vec{B}_0\cdot\vec{\nabla}(\vec{\nabla}\times\vec{B}_1) = -e\mu_0 n_0 \frac{\partial}{\partial t}\vec{B}_1$$ → Combining ① and ② → ③

Next, taking Fourier transformation (considering planar wave soln.)

$$\vec{B}_0\cdot i\vec{k}\,(i\vec{k}\times\vec{B}_1) = -e\mu_0 n_0(-i\omega)\vec{B}_1$$

$$\Rightarrow \quad B_0 k_{\parallel}(\vec{k}\times\vec{B}_1) = -ie\mu_0 n_0 \omega \vec{B}_1$$ → ④

NOTE: $\vec{B}(t,\vec{r}) \xrightarrow{\text{Fourier}} \vec{B}(\omega,\vec{k})$

From ④

To satisfy the relation, $\vec{B}_1$ has to be orthogonal to $\vec{k}$ i.e.

$$\boxed{\vec{B}_1 \cdot \vec{k} = 0}$$

Otherwise $\vec{k} \times \vec{B}_1$ would give us some vector $\neq \vec{B}_1$

We also assume $\vec{k}$ to be real and $\vec{k}$ need not be in the direction of $\vec{B}_0$.

Now the next dilemma is the existence of the equ. ④ itself. In general the expression says the fluctuation in magnetic field vector $\vec{B}_1$ is perpendicular to itself which is impossible unless it's a complex quantity. In other way, even if we consider the orthogonal relation for $\vec{B}_1$, the left hand side of equ. ④ is real when B_1 is real but the right hand side is complex. Therefore,

$$\boxed{\vec{B}_1 \rightarrow \text{Complex vector}}$$

$$\boxed{\vec{B}_1 = \vec{a} + i\vec{b}}$$ where, $\vec{a}, \vec{b}$ are real vectors

Few properties of a, b:

$$\xi = B_0 k_{\parallel} / (e \mu_0 n_0 \omega)$$

$$\xi \vec{k} \times \vec{a} = \vec{b} \qquad \xi \vec{k} \times \vec{b} = -\vec{a} \quad \rightarrow (5)$$

1) $\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$, for $\vec{k} \neq 0$

2) $\vec{k} \cdot (5) \Rightarrow \vec{k} \cdot \vec{a} = 0, \vec{k} \cdot \vec{b} = 0 \Rightarrow \vec{k} \perp \vec{a}, \vec{k} \perp \vec{b}$

3) $\xi (\vec{k} \times \vec{a}) \cdot \vec{b} = b^2, \quad \xi (\vec{k} \times \vec{b}) \cdot \vec{a} = -a^2$

$\Rightarrow a^2 = b^2, \quad a = \pm b$

(4) $|\vec{a}| = |\vec{b}|$, but they are orthogonal

(5) $\xi > 0, \vec{k} > 0 \Rightarrow \vec{k}, \vec{a}, \vec{b}$ forms basis of right hand orthogonal system.

Therefore, if we can express $\vec{B}$ as $\vec{B} = \vec{a} + i\vec{b}$

The variations in B-field can be given as,

$$\mathbb{R}\left\{ a(\hat{x} + i\hat{y}) \exp\left(-i(\omega t - \vec{k} \cdot \vec{r})\right)\right\}$$

If we assume,

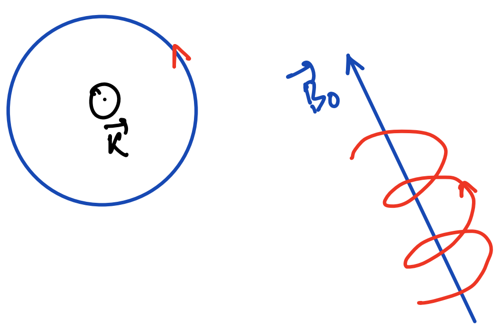
$$\boxed{\begin{aligned} \vec{a} &\to \hat{x} \\ \vec{b} &\to \hat{y} \\ \vec{k} &\to \hat{z} \end{aligned}}$$

Then, for the fluctuating magnetic field $\vec{B}_1$

$$\boxed{\mathbb{R}\left\{ a(\hat{x}+i\hat{y}) \exp\left[-i(\omega t - \vec{k}\cdot\vec{r})\right]\right\} = a\cos(\omega t - kz)\,\hat{x} + a\sin(\omega t - kz)\,\hat{y}} \rightarrow (6)$$

The expression represents a circularly polarized wave.

if $\vec{k} \parallel \vec{B}_0$



RIGHT POLARIZATION ONLY

For a representation of wave like equ. (6) the fluctuation in the magnetic field will only experience right hand rotation.

For left hand rotation, it must have a form,

$$\mathbb{R}\left\{ \vec{a}(\hat{x}-i\hat{y}) \exp\left[-i(\omega t - \vec{k}\cdot\vec{r})\right]\right\}$$

However, this does not represent electron motion instead it represents the ions.

The only possible way to get a linearly polarized wave is to superpose left and right circularly polarized waves i.e. to consider both electron and ion motion.

It is to be noted that electrons follow the same pattern i.e. the direction of rotation as they respond to a external magnetic field while they move (gyromotion).

DISPERSION RELATION:

We have, $\xi \vec{k} \times \vec{a} = \vec{b}$ $\qquad$ $\xi \vec{k} \times \vec{b} = -\vec{a}$

Now, $\xi \vec{k} \times (\vec{k} \times \vec{a}) = \vec{k} \times \vec{b}$

$$\Rightarrow \xi \vec{k} \times (\vec{k} \times \vec{a}) = \frac{1}{\xi} - \vec{a}$$

$$\Rightarrow \xi^2 \vec{k} \times (\vec{k} \times \vec{a}) = -\vec{a}$$

BAC- CAB

$$\Rightarrow \xi^2 \; \vec{k}(\vec{k} \cdot \vec{a}) - \vec{a} \cdot k^2 = -\vec{a}$$

$0 \; (\because \vec{k} \perp \vec{a})$

$$\Rightarrow \boxed{\xi^2 k^2 \vec{a} = \vec{a}} \rightarrow (7)$$

Rewriting equ. 4

$$B_0 K_{\parallel} (\vec{K} \times \vec{B_1}) = -i e \mu_0 n_0 \omega \vec{B_1}$$

Taking $\vec{K} \times$

$$B_0 K_{\parallel} \vec{K} \times (\vec{K} \times \vec{B_1}) = -i e \mu_0 n_0 \omega (\vec{K} \times \vec{B_1})$$

$$= -(e \mu_0 n_0)^2 \omega^2 \vec{B_1} \frac{1}{(B_0 K_{\parallel})}$$

$$\Rightarrow (B_0 K_{\parallel})^2 \vec{K} \times (\vec{K} \times \vec{B_1}) = -(e \mu_0 n_0)^2 \omega^2 \vec{B_1}$$

$$\Rightarrow \left(\frac{B_0 K_{\parallel}}{e \mu_0 n_0 \omega}\right)^2 \vec{K} \times (\vec{K} \times \vec{B_1}) = -\vec{B_1}$$

$$\Rightarrow \xi^2 \vec{K} \times (\vec{K} \times \vec{B_1}) = -\vec{B_1}$$

Using 7

$$\Rightarrow \xi^2 K^2 \vec{B_1} = \vec{B_1}$$

Dispersion,

$$\xi^2 K^2 = 1$$

$$\Rightarrow \left(\frac{B_0 K_{\parallel} K}{e \mu_0 n_0}\right)^2 = \omega^2$$

$$\Rightarrow \omega^2 = \left(\frac{B_0 K_{\parallel} K}{e \mu_0 n_0}\right)^2 = \left(\frac{\vec{B_0} \cdot \vec{K}}{e \mu_0 n_0} K\right)^2$$

SCALAR FORM VECTOR FORM

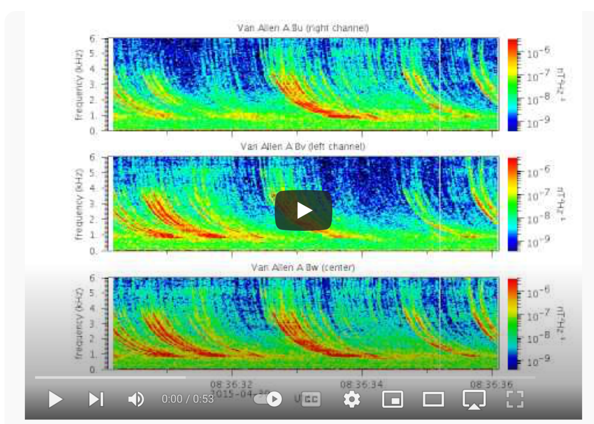
The dispersion is only valid

$$\Omega_{pi}, \Omega_{ci} \ll \omega \ll \omega_{ce}$$

The whistlers are TRASVERSE or SHEAR wave like Alfve'n waves as $\vec{J} \perp \vec{k}$

Finally, $\omega \sim k^2$ for whistlers that propagate along $\vec{B}_0$

Whistlers are usually produced in the equitorial region and they then follow the magnetic field lines and propagate towards higher latitude.



Van Allen Probes A: "Whistler Wars" 2015-04-30