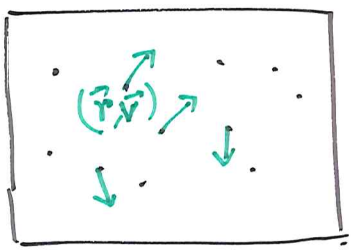
BASICS OF CONTINUUM MODELS

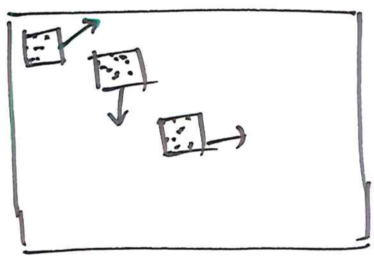
Plasma is a complex medium consisting of electrons, ions and neutrals which moves under the influence of electric, magnetic and collisional forces. To understand plasma and its basis of dynamics, we'll look into classical continuum mechanics for neutral fluid and gases.

For single particls : VECTOR QUANTITIES $(\vec{r}, \vec{v}, \vec{F})$

For Continuum models: VECTOR AND SCALAR FIELDS

Instead of tracking individual particles, we study fields which represent an ensamble of particles.





SINGLE

$(\vec{r}, \vec{v})$

$\downarrow$

$\vec{V}$ is not $f(\vec{r})$

CONTINUUM/FLUID

$(\vec{V}, \vec{E}, \vec{B}, \ldots)$

$(\rho, T,)$

$\vec{V}(\vec{r}, t)$

Velocity field is a function of space.

For continuum model, we'll consider density (ρ) instead of mass (m) unlike the single particle motion.

CHANGE IN DENSITY [$\rho(\vec{r},t)$]

$$\underbrace{\frac{\partial}{\partial t}\rho}_{\text{Time}} \qquad \underbrace{\vec{\nabla}\rho = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)}_{\text{SPACE}}\rho$$

RECALLING GAUSS THEOREM (DIVERGENCE THEOREM)

$$\boxed{\iiint_V \vec{\nabla}\cdot\underset{\downarrow}{\vec{u}}\, dV = \iint_S \vec{u}\cdot\underset{\downarrow}{\hat{n}}\, ds}$$

Field vector

surface normal

GAUSS LAW IN ELECTROSTATICS

$$\boxed{\int_S \vec{E}\cdot d\vec{s} = \frac{Q}{\epsilon_0}}$$

$$4\pi r^2 \hat{r}\, E(r) = \frac{Q}{\epsilon_0}$$

$$\boxed{\Rightarrow E(r) = \frac{Q}{4\pi\epsilon_0 r^2}}$$

using Divergence theorem

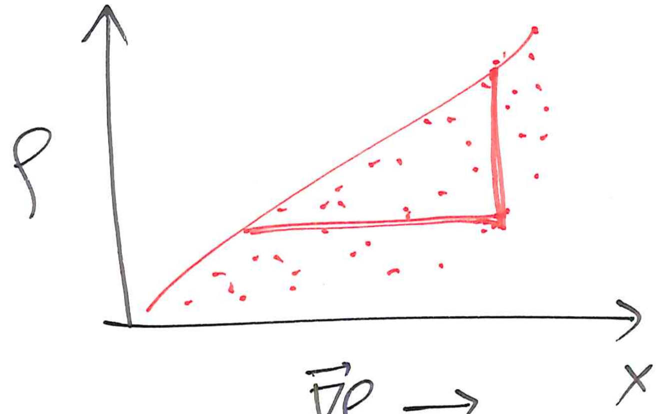
$$\boxed{\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho_e}{\epsilon_0}}$$

$$\nabla^2\phi = -\frac{\rho_e}{\epsilon_0}$$

where, $E = -\nabla\phi$

DENSITY FIELD (GENERAL FLUID)

IN 1D:



$\vec{\nabla}\rho \rightarrow$

DISCRETE APPROACH

Δ is large.

1) Density variation:

$$\rho(x+\Delta x) - \rho(x) = \Delta\rho$$

$$\Delta\rho > 0 \quad \text{or} \quad \Delta\rho < 0$$

For a stationary case where we only have spatial gradient.

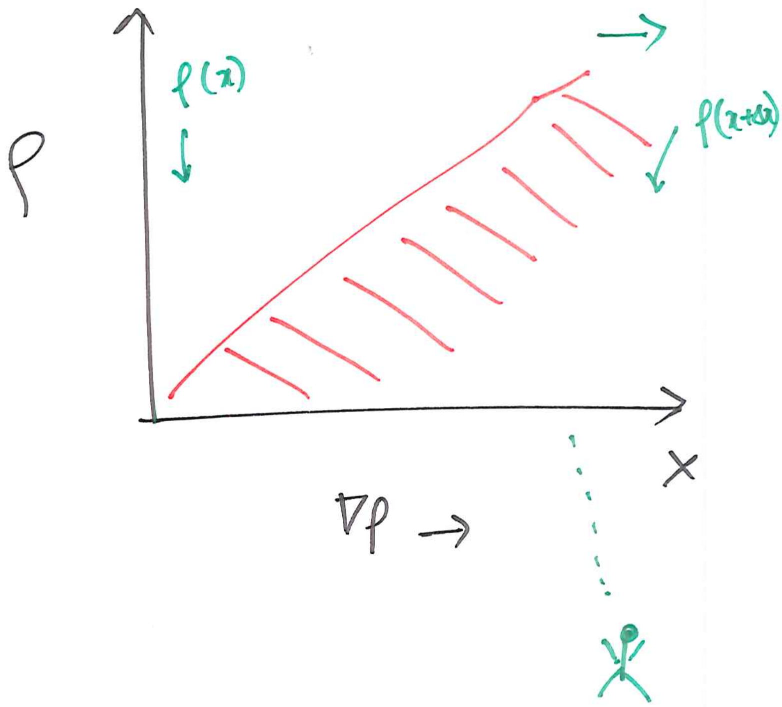
2) Dynamic density (density moving)

$$u = \frac{\Delta x}{\Delta t}$$

density element

Velocity field (Fluid flow)

FIXED POINT OBSERVER
OR
LABORATORY FRAME OF REFERENCE
OR
STANDING OBSERVER



The observer will see

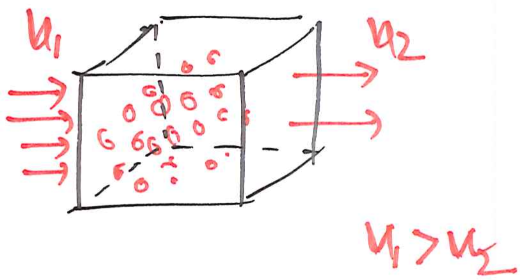
$-\Delta\rho$ in time

$$\frac{\Delta\rho}{\Delta t} = -u\frac{\Delta\rho}{\Delta x}$$

$$\boxed{\Rightarrow \quad \frac{\partial\rho}{\partial t} = \dot{\rho} = -u\frac{\partial\rho}{\partial x}}$$

FOR SITUATIONS WHEN WE HAVE CHANGE IN VELOCITY

i.e. $\vec{\nabla}u \neq 0$



means more particle packed together over time that leads to compressibility.

if more incoming flow than outgoing, there will be increase in density over time.

NOTE: For compressibility, we need not to have $\nabla \rho$. Only $\nabla u \neq 0$ should be enough.

$$u(x+\Delta x) - u(x) = \Delta u$$

For a fixed observer, $\Delta u < 0$

$$\boxed{\frac{\Delta \rho}{\Delta t} = -\rho \frac{\Delta u}{\Delta x}}$$

CONSERVATION OF MASS

REMEMBER: For a fixed observer, (incoming field − outgoing field) to decide the sign.

CONSERVATION OF MASS IN 1D

$$M = \rho \Delta x \qquad \Delta(\rho \Delta x) = (\rho u_{IN} - \rho u_{OUT}) \Delta t$$

$$\boxed{\Rightarrow \frac{\Delta \rho}{\Delta t} = -\rho \frac{\Delta u}{\Delta x}}$$

IN GENERAL,

$$\boxed{\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u)}$$

~~IN 3D:~~

IN 3D:

CONTINUITY EQN:

when there is no production or loss

$$\boxed{\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot(\vec{u}\rho) = 0}$$

Change in mass density is equal to: total change in mass flux in space.

mass density.

mass flux

Divergence of mass flux.

TAKING $\int$ volume integral $\Rightarrow$ mass conservation.

MOVING OBSERVER:

$$\frac{D}{Dt} = \boxed{\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}}$$

Convective derivative

The variation in the flow due to convection.

$$\boxed{\frac{D}{dt}\rho} = \boxed{-\rho\,\vec{\nabla}\cdot\vec{u}}$$

compressibility

Change in time + movement of the system

IF FLUID IS INCOMPRESSIBLE

$$\frac{D}{Dt}\rho = 0$$

OLD →
$$\boxed{\frac{\partial}{\partial t}\rho + \vec{u}\cdot\frac{\partial \rho}{\partial x} = 0}$$

FOR A COMOVING OBSERVER NO DENSITY GRADIENT.

IN A COMOVING FRAME, ANY CHANGES ARE DUE TO EXPANSION OR COMPRESSION OF THE FLUID.

$$\downarrow$$

$$-\rho\vec{\nabla}\cdot\vec{u}$$

IF PRODUCTION OR LOSS

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot(\vec{u}\rho) = \underbrace{\alpha}_{\text{production}} - \underbrace{\beta}_{\text{loss}}$$

For Plasma, (Charge Continuity)

$$\frac{\partial}{\partial t}(q\cdot n) + \nabla\cdot\vec{J} = 0 \qquad \vec{J} = qn\vec{u}$$

FIXED POINT OBSERVER ⟶ EULERIAN APPROACH ↘ $\frac{\partial}{\partial t}$

COMOVING OBSERVER ⟶ LAGRANGIAN APPROACH ↘ $\frac{D}{Dt}$

VELOCITY FIELD

NEWTON'S SECOND LAW → FOR SINGLE PARTICLE

$$m\vec{a} = \vec{F} = m\frac{d\vec{v}}{dt}$$

velocity of particle.

$\vec{v}(t)$

Function of time only

IN FLUID,

$$\vec{u} = \vec{u}(\vec{r}, t)$$

position

time

IN FLUID,

velocity → velocity field

mass → mass density field

$$\boxed{\rho \frac{\partial \vec{u}}{\partial t} = f}$$

Force density

$$\vec{f} = \frac{\vec{F}}{V}$$

Force per volume

This is valid if $\vec{u}$ is const./homogeneous at $t=0$

LAGRANGIAN.

$$\rho \frac{D}{Dt}\cdot\vec{u} = \rho\left(\frac{\partial}{\partial t}\vec{u} + \vec{u}\cdot\vec{\nabla}\vec{u}\right) = \vec{f}$$

CONSIDERING CHANGE IN MOMENTUM IN A SMALL VOLUME

$$\frac{d}{dt}\int_V \rho\vec{u}\,d\vec{r} = -\oint_S \rho\vec{u}\,\vec{u}\cdot\hat{n}\,ds + \int_V \vec{f}\,d\vec{r}$$

using GAUSS THEOREM

$$= \int_V \left[\frac{\partial}{\partial t}(\rho \vec{u}) + \nabla \cdot \rho \vec{u}\vec{u} \right] d\vec{r}$$

$$= \int_V \left[\vec{u} \frac{\partial}{\partial t}\rho + \rho \frac{\partial}{\partial t}\vec{u} + \nabla(\rho\vec{u})\vec{u} + (\rho\vec{u})\cdot(\nabla\vec{u}) \right] d\vec{r}$$

CONTINUITY EQN $\longrightarrow$ For no production or loss $\rightarrow 0$

$$= \int_V \left[\rho \frac{\partial \vec{u}}{\partial t} + (\rho\vec{u})\cdot\nabla\vec{u} \right] d\vec{r}$$

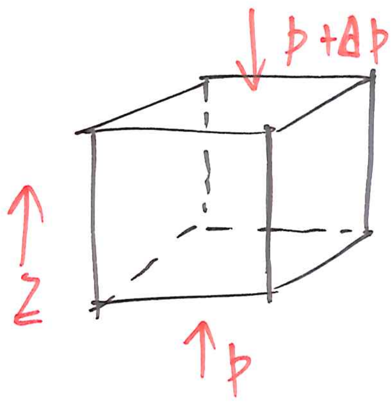
$$\Rightarrow \int_V \rho \left(\frac{\partial}{\partial t}\vec{u} + \vec{u}\cdot\nabla\vec{u} \right) d\vec{r} = \int_V \vec{f}\, d\vec{r}$$

CONTINUITY MOMENTUM

AT THIS POINT, WE HAVE TWO EQNS AND THREE UNKNOWNS $(\rho, \vec{u}, \vec{f})$

Force $\vec{f}$ can be represented as

$$\vec{f} = -\vec{\nabla} p$$



Force

$$F = A \cdot \left(p(z) - p(z + \Delta z) \right)$$

$$\approx -A \cdot \Delta z \frac{\Delta p}{\Delta z}$$

$$\frac{F}{A \cdot \Delta z} = -\frac{\Delta p}{\Delta z} = -\nabla p$$

$$\Rightarrow f = -\vec{\nabla} p$$

THE MOST RELEVANT FORM OF MOMENTUM EQN FOR FLUID

$$\rho \left(\frac{\partial}{\partial t} \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} \right) = -\nabla p + \mu \nabla^2 \vec{u} + f$$

pressure viscosity

NEXT → EQN OF STATE