DYNAMIC SOLUTIONS OF FLUID EQN

(1) CONTINUITY EQN

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

(2) MOMENTUM EQN.

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right) = -\nabla p$$

(3) EQN. OF STATE

$$p = f(\rho, T)$$

STEADY STATE SOLUTIONS

$\vec{u} = 0$ (could be const. as well) TRICK: FRAME OF REF.

$p = p_0 = \text{Const.}$

$\rho = \rho_0 = \text{Const.}$

DYNAMIC SOLUTIONS

$\vec{u} = 0 + \vec{u}_1$ — very small; first order perturbation

$p = p_0 + p_1$

$\rho = \rho_0 + \rho_1$

$$\boxed{p_1 \ll p_0 \qquad \rho_1 \ll \rho_0}$$

Using dynamic solutions,

$$(1) \Rightarrow \frac{\partial}{\partial t}(\rho_0+\rho_1) + \vec{\nabla}\cdot[(\rho_0+\rho_1)\vec{u}_1] = 0$$

(Process is referred as LINEARIZATION)

$$\Rightarrow \cancelto{0}{\frac{\partial}{\partial t}\rho_0} + \frac{\partial}{\partial t}\rho_1 + \vec{\nabla}\cdot(\rho_0\vec{u}_1) + \cancelto{0}{\nabla\cdot(\underbrace{\rho_1}_{\text{small}}\underbrace{\vec{u}_1}_{\text{small}})} = 0$$

small × small → very small

$$\Rightarrow \boxed{\frac{\partial \rho_1}{\partial t} + \vec{\nabla}\cdot(\rho_0\vec{u}_1) = 0} \qquad (1\cdot A)$$

NOTE: If ρ_1 or u_1 is big, we can't apply LINEARIZATION.

(2) $\Rightarrow$ $$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right) = -\vec{\nabla} p$$

$$\Rightarrow (\rho_0+\rho_1)\left[\frac{\partial \vec{u}_1}{\partial t} + \vec{u}_1\cdot\vec{\nabla}\vec{u}_1\right] = -\vec{\nabla}(p_0+p_1)$$

(LINEARIZATION) (LINEARIZATION)

$$\Rightarrow \rho_0\frac{\partial \vec{u}_1}{\partial t} + \cancel{\rho_1\frac{\partial \vec{u}_1}{\partial t}}^{\,0} + \cancel{\rho_1(\vec{u}_1\cdot\vec{\nabla}\vec{u}_1)}^{\,0} = -\cancel{\vec{\nabla} p_0}^{\,0} - \vec{\nabla} p_1$$

SMALL SMALL SMALL

$$\Rightarrow \boxed{\frac{\partial \vec{u}_1}{\partial t} = -\frac{\vec{\nabla} p_1}{\rho_0}} \quad (2\cdot A)$$

(3) $\Rightarrow$ $$p = p(\rho)$$

For small perturbation in density, we can approximate the order perturbation in pressure as,

$$\boxed{p_1 = \left.\frac{dp}{d\rho}\right|_{\rho_0}\rho_1} \quad (3\cdot A)$$

Now, let's rewrite the continuity eqn. (1·A)

$$\frac{\partial \rho_1}{\partial t} + \vec{\nabla}\cdot(\rho_0\vec{u}_1) = 0$$

$$\Rightarrow \frac{\partial \rho_1}{\partial t} + \rho_0\vec{\nabla}\cdot\vec{u}_1 + \vec{u}_1\cdot\vec{\nabla}\rho_0 = 0$$

$$\Rightarrow \boxed{\frac{\partial \rho_1}{\partial t} + \rho_0\vec{\nabla}\cdot\vec{u}_1 = 0} \quad (1\cdot B)$$

LET'S START COMBINING

(1.B), (2.A), and (3.A)

FIRST:

Multiply (1.B) with $\left.\frac{dp}{d\rho}\right|_{\rho_0}$

$$\left.\frac{dp}{d\rho}\right|_{\rho_0} \frac{\partial \rho_1}{\partial t} + \left.\frac{dp}{d\rho}\right|_{\rho_0} \rho_0 \vec{\nabla}\cdot\vec{u}_1 = 0$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial t} \left.\frac{dp}{d\rho}\right|_{\rho_0} \rho_1}_{\text{EQN. OF STATE}} + \rho_0 \left.\frac{dp}{d\rho}\right|_{\rho_0} \vec{\nabla}\cdot\vec{u}_1 = 0$$

$$\boxed{\Rightarrow \frac{\partial}{\partial t} p_1 + \rho_0 \left.\frac{dp}{d\rho}\right|_{\rho_0} \vec{\nabla}\cdot\vec{u}_1 = 0} \quad (4)$$

NOW, LET'S INTRODUCE VELOCITY POTENTIAL

$$\vec{u} = \vec{\nabla}\psi$$ VELOCITY POTENTIAL

THIS ALLOW US TO IGNORE THE ROTATIONAL OR SHEARED VELOCITY VARIATION.

$$\vec{\nabla}\times\vec{u} = \vec{\nabla}\times\vec{\nabla}\psi = 0$$

LET'S USE THE VELOCITY POTENTIAL IN MOMENTUM EQN. (2.A)

$$\rho_0 \frac{\partial \vec{u}_1}{\partial t} \Rightarrow \rho_0 \frac{\partial}{\partial t} \underline{\underline{\nabla}} \psi = -\underline{\underline{\nabla}} p_1$$

Allows us to remove gradient on both sides.

$$\boxed{\Rightarrow p_1 = -\rho_0 \frac{\partial \psi}{\partial t}}$$

LET'S TAKE TIME DERIVATIVE

$$\boxed{\frac{\partial}{\partial t} p_1} = -\rho_0 \frac{\partial^2 \psi}{\partial t^2}$$

MODIFIED (EQN. OF STATE + CONTINUITY EQN) (4)

$$\Rightarrow -\rho_0 \left.\frac{dp}{d\rho}\right|_{\rho_0} \vec{\nabla} \cdot \vec{u}_1 = -\rho_0 \frac{\partial^2 \psi}{\partial t^2}$$

$$\Rightarrow -\rho_0 \frac{\partial^2 \psi}{\partial t^2} + \rho_0 \left.\frac{dp}{d\rho}\right|_{\rho_0} \nabla \cdot \nabla \psi = 0$$

$$\boxed{\Rightarrow \frac{\partial^2 \psi}{\partial t^2} - \left.\frac{dp}{d\rho}\right|_{\rho_0} \nabla^2 \psi = 0} \quad (5)$$

THE VARIATION OF THE PRESSURE CAN BE DEFINED AS,

$$\left.\frac{dp}{d\rho}\right|_{\rho_0} = c_s^2$$, where c_s = sound speed

$$\Rightarrow c_s = \sqrt{\left.\frac{dp}{d\rho}\right|_{\rho_0}}$$

5 $\Rightarrow$

$$\boxed{\frac{\partial^2 \psi}{\partial t^2} - c_s^2 \nabla^2 \psi = 0} \quad (6)$$

NOW, LET'S REVISIT c_s

NATURAL CHOICE FOR NEUTRAL GASES IS ADIABETIC EQN OF STATE. i.e. flow is faster than the heat spread

$$p = p_0 \left(\frac{\rho}{\rho_0}\right)^\gamma$$

$$\frac{dp}{d\rho} = \gamma p_0 \frac{\rho^{\gamma-1}}{\rho_0^\gamma} = \gamma \underbrace{p_0 \left(\frac{\rho}{\rho_0}\right)^\gamma}_{p} \cdot \frac{1}{\rho}$$

$$\Rightarrow \frac{dp}{d\rho} = \gamma \frac{p}{\rho}$$

$$\cancel{\left.\Rightarrow \frac{dp}{d\rho}\right|_{\rho_0}}$$

At $\rho = \rho_0$

$$p = nKT$$

Finally, ②

$$\frac{dp}{d\rho} = \gamma \frac{nKT}{\rho}$$

$$= \gamma \frac{\rho KT}{M\rho}$$

$$\Rightarrow \boxed{\frac{dp}{d\rho} = \gamma \frac{KT}{M}}$$

For ideal gases, $\gamma = 5/3$

$$\Rightarrow C_s = \sqrt{\gamma \frac{KT}{M}} = \sqrt{\frac{5}{3} \frac{KT}{M}}$$

$$\Rightarrow \boxed{C_s = \sqrt{\frac{5}{3} \frac{KT}{M}}}$$

SOUND SPEED

Now, the wave eqn.

$$\boxed{\frac{\partial^2 \psi}{\partial t^2} - c_s^2 \nabla^2 \psi = 0}$$

Can be solved taking Fourier transformation assuming planar wave solution,

$$\boxed{\psi = \psi_0 \exp\left[-i(\omega t - k\vec{r})\right]}$$

$$\frac{\partial \psi}{\partial t} = -i\omega\psi \qquad \frac{\partial^2 \psi}{\partial t^2} = -i\omega \cdot -i\omega\psi$$
$$= -\omega^2 \psi$$

$$\nabla \psi = ik\psi \qquad \nabla^2 \psi = ik \cdot ik\psi$$
$$= -k^2 \psi$$

Now, put these back into the wave eqn.

$$-\omega^2 \psi + c_s^2 k^2 \psi = 0$$

$$\boxed{\Rightarrow \omega^2 = c_s^2 k^2} \qquad (\psi \neq 0)$$

DISPERSION RELATION → $c_s = \frac{\omega}{k}$

does not

$$\frac{d\omega}{dk} = \frac{\omega}{k} = \text{Const.}$$

→ NON DISPERSIVE MEDIUM